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ORIGINAL ARTICLE

How to make ‘more’ better? Principles for effective use of multiple representations to enhance students’ learning about fractions

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Abstract To make complex mathematics concepts accessible to students, teachers often rely on visual representations. Because no single representation can depict all aspects of a mathematics concept, instruction typically uses multiple representations. Much research shows that multiple representations can have immense benefits for students’ learning. However, some re-search also cautions that multiple representations may fail to enhance students’ learning if they are not used in the “right” way. For example, unless students can (1) properly interpret each individual representation and (2) make connections among multiple representations and the information they intend to convey, the use of multiple representations may actually confuse students rather than aid their learning. In this article, we review research-based principles for how to use multiple representations effectively so that they enhance student learning. Using fractions as an illustrative domain, we discuss how the choice of visual representation may affect student learning based on the conceptual aspects of the to-be-learned content emphasized by the representation. Next, we describe ways to help students interpret individual representations and to make connections among them. We illustrate these arguments with our own empirical research on fractions learning. We conclude by laying out open questions that future research should address.

Keywords Fractions · Multiple representations · Conceptual learning · Perceptual learning

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1 Introduction

We often use visual representations to support mathematical thinking. Data visualizations are used in statistics, function graphs in algebra, and schematic drawings in geometry—the examples are boundless. Indeed, abundant theoretical work [e.g., Ainsworth 2006; National Research Council (NRC), 2006; Schnotz 2005; Uttal and O’Doherty 2008] as well as empirical work (see Rau 2016 for a summary) supports this practice, suggesting that adding visual representations (e.g., graphs and other illustrations) to text-based representations (e.g., word problems) and symbolic representations (e.g., equations) can enhance students’ learning.

Because no single visual representation perfectly depicts the complexity of mathematical concepts, instructors often use *multiple* visual representations, where the different representations emphasize complementary conceptual aspects. For example, students learning about fractions typically encounter visual representations like those in Fig. 1: circle diagrams show fractions as parts of an inherent whole, whereas number lines show fractions as measures of relative magnitude (Kieren 1993; Post et al. 1982). Instructors may use visual representations (1) to familiarize students with visual conventions commonly used in the mathematics community, (2) to illustrate abstract complex concepts, (3) to enlarge the set of tools students have for engaging mathematics, and (4) to leverage students’ subjective preferences and expertise (see Acevedo Nistal et al. 2009; Singer 2007).

However, multiple visual representations are not always more effective for promoting learning (Rau et al. 2015). It is critical to remember that—even if a visual representation is more concrete than a symbolic representation—the visual representation nevertheless remains something

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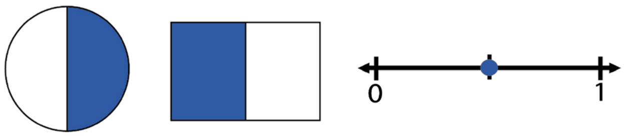


Fig. 1 Visual representations commonly used in fractions instruction: *circle*, *rectangle*, and *number line*

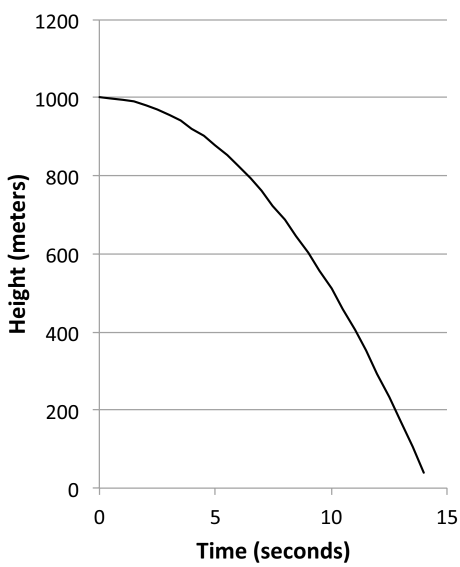


Fig. 2 Graph of height versus time for a rock dropped from a height of 1000 m under the force of gravity

that stands for a referent (i.e., what the representation is meant to depict) in the mathematical domain. Unfortunately, relations between visual representations and their referents are often opaque, leading to difficulties learning with them (e.g., Ainsworth et al. 2002; diSessa 2004; NRC 2006). These difficulties are implicated in the *representation dilemma* (Rau 2016): On the one hand, students have to learn domain knowledge *from* visual representations. For example, students can learn *from* the line graph in Fig. 2 that the distance a rock dropped off a cliff travels in a given time interval increases as time increases. On the other hand, students have to learn *about* the visual representations; namely about its relationship to the referent. In learning *about* the graph in Fig. 2, they must learn what each axis represents and how to coordinate between the two. The representation dilemma presents a major educational challenge: Students are often expected to use unfamiliar visual representations to learn about unfamiliar concepts.

To overcome this dilemma, students need *representational competencies*: knowledge and skills that enable them to use visual representations to reason about and solve tasks (NRC 2006). The goal of this article is to present instructional principles that can help students acquire representational competencies while they learn domain knowledge, thereby ensuring that 'more' (multiple) visual representations lead to better learning outcomes. This article seeks to close a gap between prior research on learning with visual representations that has, by and large, focused on learning with text and a single type of visual representation (see Rau 2016), and the recommendation by mathematics practice guides to use multiple visual representations [National Council for Teachers of Mathematics (NCTM) 2000, 2006; National Mathematics Advisory Panel (NMAP) 2008; Siegler et al. 2010]. To close this gap, we review extant research on representational competencies and summarize principles that can guide instruction in accounting for students' representational competencies.

Although the principles we present should apply broadly, in this article, we chose the domain of fractions for illustrative purposes. We chose fractions for two reasons: First, recent literature has detailed how critical fractions knowledge is for ensuring success in later mathematics (e.g., Siegler et al. 2011). Second, fractions represents a complex domain with many aspects that are differentially highlighted by alternate choices of representation (Behr et al. 1993; Ohlsson 1988). As such, thorough understanding can only be reached by use of multiple representations which support learning of complementary aspects of the domain. To support our argument, we rely heavily on the first author's work with the Fractions Tutor: an intelligent tutoring system for fractions learning that focuses on supporting students' representational competencies (e.g., Rau et al. 2015).

2 Why multiple representations: complementary conceptual advantages of visual representations of fractions

The mathematics education literature suggests that visual representations fundamentally shape how students conceptualize fractions (Charalambous and Pitta-Pantazi 2007). Moreover, instruction tends to use a multiplicity of representations of fractions—in part because fractions are a notoriously complex domain. Knowledge of fractions and rational numbers can be seen as a "mega-concept" composed of several subconstructs or alternative interpretations (Behr et al. 1993; Ohlsson 1988). For instance, Behr and colleagues (1993) suggest at least six conceptual ways to interpret fractions: (1) parts of a whole, (2) decimals, (3) ratios, (4) quotient, (5) operators, and (6) measurements.

Visual representations appear to differ in their capacity to help students understand different aspects of fractions knowledge (Charalambous and Pitta-Pantazi 2007; Kieren 1993). Arguments for the merits and limitations of various visual representations for fractions are abound: Cramer and Wyberg (2009) argued discrete models, such as chip models (see Fig. 3d) may be well-suited to illustrate part-whole concepts (e.g., one of four sections is shaded), but may not be well-suited for helping children understand fractions as numbers with specific magnitudes. On the other hand, they found that area representations (e.g., circles, rectangles) can help students understand the role the denominator for the size of a fraction, but are less helpful in supporting estimation with fractions. Some (e.g., Cramer and Wyberg 2009) have argued that circle representations are the most powerful concrete representation because the whole circle inherently serves as a unit while illustrating both part-whole concepts and the meaning of the relative sizes of fractions. Siegler and colleagues (2011) have argued that number lines are most powerful because they help support an integrated model of numerical magnitude that unites fractions with all real numbers. Others have argued that vector representations—specifically their slopes—are particularly efficient at helping students understand the relational nature of the way fractions concatenate numerators and denominators (Ohlsson 1988).

In sum, even these cursory reflections on the vast literature on fractions learning reveal that no single visual representation can convey the multiplicity of *related but only partially overlapping ideas* that constitute the fractions mega-concept (Ohlsson 1988). Failure to make connections among these representations may lead students to rely exclusively on one conceptual interpretation, which can constrain understanding (Behr et al. 1993; Kieren 1993). An example of a prominent misconception is the whole number bias. Reliance on symbolic representations of fractions is associated with the whole number bias, whereby students misapply concepts and procedures that are appropriate for whole numbers but inappropriate for fractions (see Ni and Zhou 2005 for a review). For example, Mack (1995) found that 6th -grade students typically claimed that 1/8 was greater than 1/6 because 8 is greater than 6. Boyer et al. (2008) found a similar whole number bias for 4th-grade students when comparing visual representations of fractions composed of partitioned bars. This bias was much less prevalent when comparisons were presented with unpartitioned bars than with partitioned bars. Therefore, it has been argued that heavy reliance of instruction on visual representations with easily countable pieces may be suboptimal in helping students overcome the whole number bias (Lewis et al. 2015). Consequently, exposing students to a variety of visual representations may help them understand abstract concepts (Singer 2009). Therefore, fractions instruction needs to help students take advantage of multiple visual representations to build robust understanding without being confused by the multitude of perspectives.

3 Representational competencies

Prior research on learning with multiple representations in a variety of domains suggests that students can only take advantage multiple visual representations if they acquire a number of representational competencies (for an overview see Rau 2016). Table 1 provides a summary of these representational competencies (discussed in this

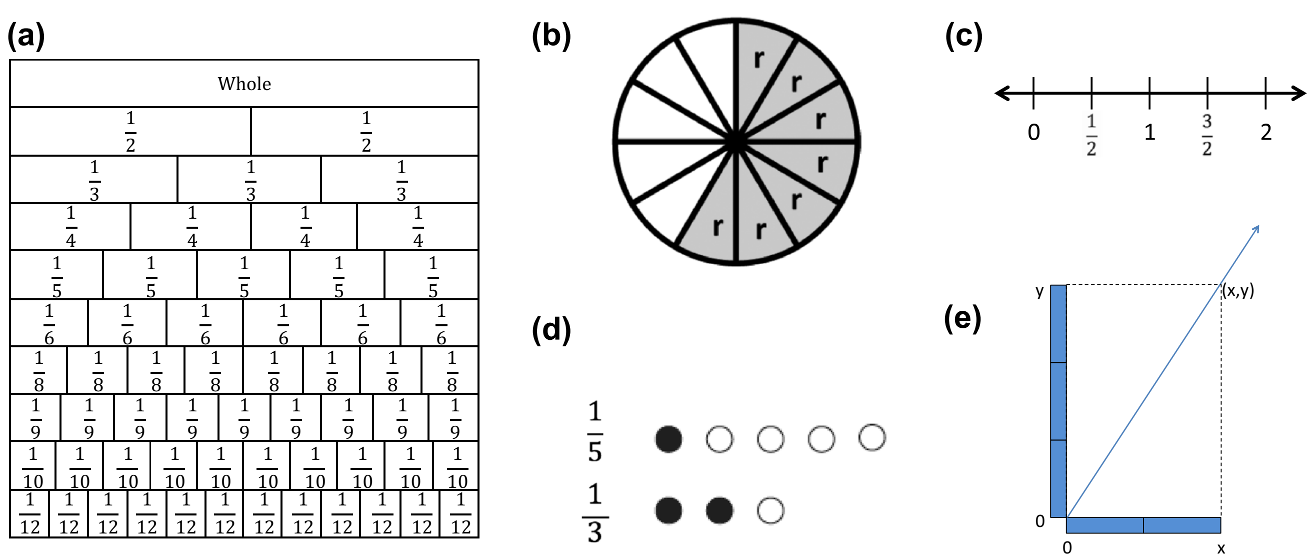


Fig. 3 Some typically used visual representations include **a** fraction bars/area models, **b** circle representations/pie charts, **c** number lines, **d** discrete chip models, **e** vector representations, **a**–**d** were adapted from Cramer et al. (2008)

Table 1 Overview of representational competencies and principles for the design of instructional supports

Representational competences	Involved knowledge and skills	Learning principles
Visual understanding	Ability to connect visual representation to concepts Ability to distinguish relevant and irrelevant visual features	#1: prompt students to explain how each visual representation depicts concepts Self-explanation prompts (design guideline 1.1) Actively establish mappings (design guideline 1.2)
Visual fluency	Efficiency in seeing meaning in visual representations Access to perceptual chunks	#2: interleave visual representations and activity types Variety of representations (design guideline 2.1) Frequently switch between activity types (design guideline 2.2) Frequently switch between representations (design guideline 2.3)
Connectional understanding	Ability to connect multiple visual representations to one another Ability to explain similarities and differences between visual representations	#3: prompt students to explain mappings between multiple visual representation Verbally explain how visual features of different representations depict corresponding and complementary concepts (design guideline 3.1) Actively establish mappings (design guideline 3.2) Provide assistance (design guideline 3.3)
Connectional fluency	Efficiency in connecting multiple visual representations Access to multiple perceptual chunks Flexibility to switch between multiple visual representations	#4: expose students to many opportunities to translate among visual representations Ask students to discriminate and categorize visual representations (design guideline 4.1) Provide immediate feedback (design guideline 4.2) Variety of representations (design guideline 4.3)

section) and of design principles for instructional interventions that can support these competencies (Sect. 4).

3.1 Visual understanding

Visual understanding is the ability to make sense of a visual representation by mapping its visual features to information relevant to understanding the target domain (Ainsworth 2006; Schnotz 2005). Learners are more likely to establish mappings between visual representations and referents they show if the representations' visual features look similar to their referents (Gentner and Markman 1997). However, similarity-based mappings may not necessarily be correct or relevant. Therefore, acquiring visual understanding involves learning principles that allow students to distinguish conceptually relevant from irrelevant visual similarities between the representation and the referent. For example, students using the circle diagram in Fig. 1 may learn the general principle that circle diagrams depict fractions as the ratio of the shaded area relative to the whole area, which can be quantified as the number of equally-sized shaded sections relative to the total number of sections. A relevant feature is that the sections are equally sized, whereas an irrelevant feature is the specific color of the sections. The student may then apply this understanding to circle diagrams using different colors.

3.2 Visual fluency

Students also need to acquire *visual fluency* with each individual visual representation. Visual fluency is the ability to quickly "see" what a visual representation shows with minimal mental effort (Airey and Linder 2009; Kellman and Massey 2013). Domain experts exhibit such fluency, perceiving information more efficiently than novices (e.g., Chi et al. 1981). Thus, experts can often "see at a glance" what a representation shows without any perceived mental effort. Developing visual fluency frees cognitive resources, allowing students to engage in higher-order conceptual thinking (Gibson 2000; Kellman and Massey 2013). Visual fluency involves high efficiency in recognizing perceptual patterns (Kellman and Massey 2013) that results from perceptual chunking. Relevant visual features activate a corresponding schema from long-term memory that constitutes the relevant concepts (Taber 2013). Thus, visual fluency allows students to make direct one-on-one mappings between perceptual chunks and concepts (Koedinger et al. 2012).

3.3 Connectional understanding

Third, students need to acquire *connectional understanding*: the ability to make conceptual connections between multiple visual representations (Ainsworth 2006). Connection making involves mapping corresponding features across different visual representations (Schnotz 2005). As mentioned, visual representations use features that may be somewhat similar to the referent. Hence, students may make connections based on similarity-based mappings. However, not all of these similarities are conceptually relevant. Some of them are incidental similarities (i.e., surface similarities, such as the fact that both the circle and the rectangle in Fig. 1 are shaded). When relevant, these similarities can make it easier for students to make connections (DeLoache 2000; Gentner and Markman 1997); but if these similarities are not conceptually relevant, they may lead students to make incorrect connections. Therefore, teaching connectional understanding involves training students to distinguish surface-level similarities from conceptually relevant similarities among visual representations. Connectional understanding also involves the ability to make sense of how one visual representation can constrain the interpretation of the second representation, and how they complement one another in depicting information (Ainsworth 2006; Rau 2016). That is, students learn to explain correspondences between visual representations (i.e., how both show similar information about domain-relevant concepts), and differences between them (i.e., how they show different information about the concepts).

3.4 Connectional fluency

Fourth, students need to acquire *connectional fluency*: the ability to perceive common patterns across different visual representations, to induce information from multiple visual representations without considerable mental effort, and to flexibly translate between visual representations (Airey and Linder 2009; Kellman and Massey 2013). For example, a fractions expert might see "at a glance" that the circle in Fig. 1 and the number line in Fig. 1 show about 1/2 of some unit. That is, the fractions expert is highly fluent in perceiving information in multiple visual representations and in seeing patterns they have in common. Connectional fluency is analogous to visual fluency (Kellman and Massey 2013): Students gain *efficiency* by learning to treat the entire analog internal representation as one perceptual chunk instead of mapping particular analog features to one another. Hence, connectional fluency means that students can make simple one-on-one mappings between perceptual chunks obtained from different visual representations.

4 How to support students' representational competencies through instructional interventions?

We have investigated principles that describe effective designs of instructional interventions that help students acquire the representational competencies just described in the context of fractions learning (see Table 1 for a summary).

4.1 Supporting visual understanding: prompt students to explain how each visual representation depicts concepts

According to cognitive theories of learning, the acquisition of sense-making competencies engages a specific type of learning process: namely, sense-making processes (Koedinger et al. 2012; Rau 2016). Sense-making processes are typically verbally mediated because they involve explanations of principles by which visual representations depict conceptually relevant information (Koedinger et al. 2012). They are explicit in that students have to willfully engage in them (diSessa and Sherin 2000). To help students engage in sense-making processes, instructional interventions should ask students to reason about how the given visual representation depicts information. Hence, ***learning principle 1*** suggests that instruction should engage students in explicit sense-making processes aimed at explaining how each visual representation depicts concepts.

Prior research in a variety of domains has investigated how best to design instructional materials so as to enhance visual understanding. Two ***design guidelines*** emerge from this research. First, prompting students to self-explain how the visual representation corresponds to text-based explanations has been found to be effective (Ainsworth and Loizou 2003; Seufert 2003) (guideline 1.1). To this end, students may be asked to self-explain what concept a specific visual feature depicts (e.g., the number of shaded sections in a circle diagram depict the numerator of a fraction), to explain how the visual representation shows domain-relevant concepts (e.g., circle diagrams show that a fraction is a portion of a unit), or to fill in information from a visual representation into a text-based version of the same information (e.g., translating the information shown in a circle diagram into a word problem in which two people share a pizza). Self-explanation prompts have been shown to be particularly effective if they help students focus on "why"-questions, because this may elicit self-explanations of principled knowledge (Berthold and Renkl 2009). Further, self-explanation prompts are more effective if they ask students to self-explain specific connections than if they are open-ended (Berthold et al. 2008; van der Meij and de Jong 2011).

A second design guideline regarding the support of visual understanding is that students should actively establish the relations between the visual representation and the concepts they show by themselves (guideline 1.2). Research shows that students show higher learning outcomes if they actively create mappings between visual representations and text describing the target concepts than if they receive pre-made mappings (Bodemer and Faust 2006).

Figure 4 shows an example of how these design guidelines can be implemented in instructional materials for fractions learning. This example is taken from the Fractions Tutor, the intelligent tutoring system for fractions learning mentioned above (Rau et al. 2015). In this example, students are prompted to self-explain how number lines show fractions principles (e.g., the more sections a number line is partitioned into, the smaller the sections become, which explains the inverse relationship between the size of the denominator and the fraction's magnitude). Further, students are asked to actively establish these mappings themselves because they have to fill in gaps to respond to self-explanation prompts.

Rau and colleagues (2015) evaluated the effectiveness of such self-explanation prompts with the Fractions Tutor. 6th-grade students ($N=112$) were randomly assigned to work with one of four versions of the Tutor: (1) one using a single visual representation (namely, number lines) but no self-explanation prompts, (2) one using multiple visual representations (number lines, circles, rectangles, etc.) but no self-explanation prompts, (3) one using a single visual representation with prompts to self-explain connections between the visual representation and concepts, and (4) one with multiple visual representations and self-explanation prompts. In the versions with multiple visual representations, only one visual representation was provided at a time, but different ones over a sequence of problem-solving activities. Results showed that using multiple visual representations enhanced students' performance on a test of fractions knowledge compared to individual representations—but only if they were combined with self-explanation prompts. Put differently, without self-explanation prompts, students did not benefit from the multiplicity of visual representations.

In sum, instructors may implement learning principle 1 by following the two design guidelines reviewed above. For example, they may ask students to explain how each visual representation depicts key concepts (e.g., numerator, denominator) or to map visual features of the representation to the concepts. Because students often struggle in making such mappings, instructors should check whether students make correct mappings and provide adequate feedback and assistance.

4.2 Supporting visual fluency: interleave a variety of visual representations and activity types

Students acquire visual fluency by engaging in implicit, non-verbal, inductive processes (Koedinger et al. 2012). Fluency-building processes are inductive because they involve learning from experience with many examples without explicit instruction (Koedinger et al. 2012). These

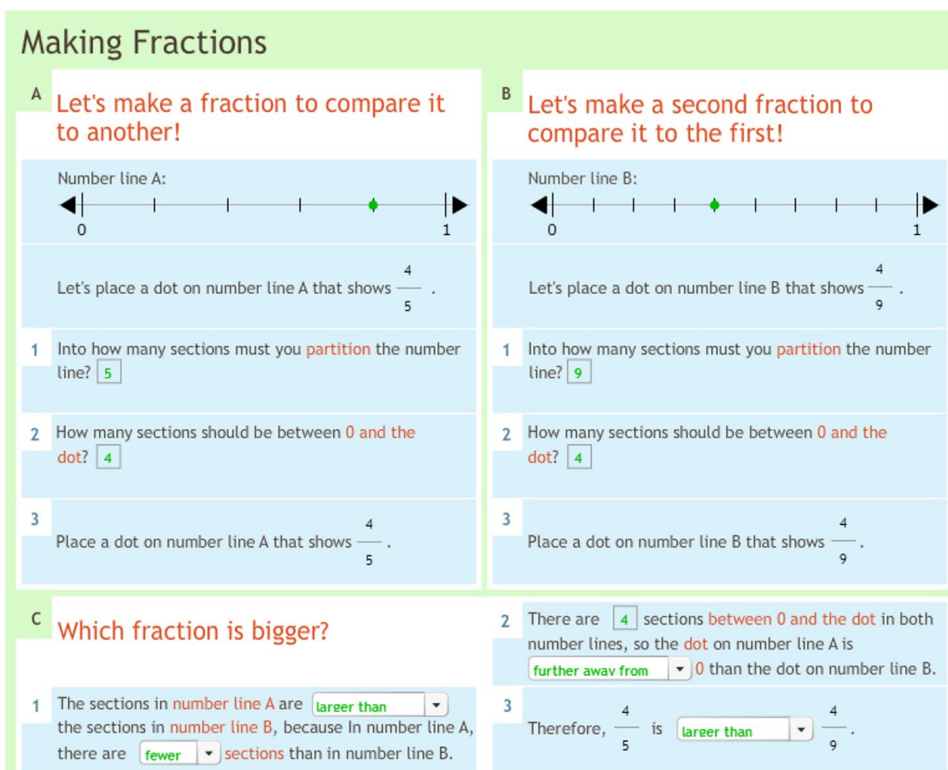


Fig. 4 Example activity from the Fractions Tutor to support visual understanding

processes are considered to be non-verbal because they do not require explicit reasoning (Kellman and Massey 2013). They are implicit because the learning processes are unintentional and unconscious (Shanks 2005), emerging from experience with many examples (Gibson 2000; Kellman and Massey 2013). Hence, ***learning principle 2*** suggests that instruction should engage students in implicit fluency-building processes aimed at learning from exposure to various example representations. We present three ***design guidelines*** that can help instructors engage students in visual fluency-building processes.

First, students should be exposed to numerous examples of the same type of visual representation, while incidental features vary and conceptually relevant features remain constant (Massey et al. 2011) (guideline 2.1). This guideline draws on research on the interference effect (de Croock et al. 1998), which suggests that interleaving aspects of a learning task that vary irrelevant features improves students' understanding of relevant features. This attuning to relevant features of representations does not require explicit, verbally mediated processes but can happen via implicit, non-verbal forms of learning (Koedinger et al. 2012).

Second, students should frequently switch between different activities for which the given type of visual representation is used (Rau et al. 2013) (guideline 2.2). This guideline also draws on the interference effect: if students encounter the visual representation across a variety of activity types for which the given representation provides useful information, students are likely to reactivate their knowledge about which visual features are conceptually relevant frequently. This process of repeated reactivation enhances the likelihood that they will be able to recall this knowledge more easily later on, without having to invest mental effort in doing so (de Croock et al. 1998; Sweller 1990).

Third, students should frequently switch between different types of visual representations (Rau et al. 2014) (guideline 2.3). Each switch requires that students reactivate their knowledge about the visual features of a specific visual representation, thereby increasing the chance that students will be able to quickly access that knowledge later on.

Our research addressed the practical question of how to combine or prioritize these principles. Specifically, when forced to make a decision, is it more important to interleave the types of activities or to interleave the types of representations? One experiment with 5th- and 6th-grade students ($N = 158$) tested whether switching frequently between visual representations is more important than frequently switching between activity types, or vice versa (Rau et al. 2013). The experiment compared a version of the Fractions Tutor in which students frequently switched between visual representations but switched less frequently between activity types (e.g., a block of successive fraction conversions using circle, rectangle, and number line representations; followed by a block of addition using the three representations; followed by a block of subtraction using the three representations) to a version in which students frequently switched between activity types but not between visual representations (e.g., a block of circle activities on fraction conversion, addition, and subtraction; followed by a block of rectangle activities on the three activity types; followed by a block of number line activities on the three activity types). Students showed higher learning outcomes if they had frequently switched between activity types rather than visual representations. This result suggests that when instructional materials involve multiple activity types and multiple visual representations, it is more important to interleave activity types (guideline 2.2) than visual representations (guideline 2.3).

A follow-up experiment with 4th- and 5th-grade students ($N = 230$) tested whether—in addition to interleaving activity types—the types of visual representations should also be interleaved (Rau et al. 2014). The experiment compared four versions of the Fractions Tutor that switched between visual representations after each activity, after every six activities, after 36 activities, or with gradually increasing frequency. All versions frequently switched between activity types. Students showed higher learning outcomes when they switched visual representations after every activity, compared to the other sequences. This result suggests that visual representations should be interleaved (guideline 2.3) in addition to activity types (guideline 2.2).

Many curricula implement learning principle 2 by "spiraling" through (i.e., switching frequently between) different topics (Harden and Stamper 1999), but they do not necessarily switch frequently between visual representations. Instructors may consider altering the sequence of problem-solving activities so that both activity types and visual representations are interleaved.

4.3 Supporting connectional understanding: prompt students to explain mappings between multiple visual representations

Similar to visual understanding, connectional understanding is the result of verbally mediated sense-making processes. Students acquire connectional understanding by engaging in conceptual, verbally mediated sense-making processes involved in relating visual features of one representation to visual features of another representation because they show corresponding conceptual aspects about the domain knowledge. The processes involved in acquiring connectional understanding are similar to the processes engaged when acquiring visual understanding, because in both cases, students explain mappings between conceptually relevant features. However, the nature of mappings is different when students make connections between multiple visual representations because they involve mappings between multiple visual features, whereas the mappings discussed above are between one visual feature and a textual or symbolic feature or an abstract concept. When students make connections between multiple visual representations, the representations may share visual features that are conceptually relevant, but they may also share irrelevant surface-level features, which may lead students to make incorrect connections. Therefore, a student's ability to make connections between multiple visual representations may build, at least to some extent, on the student's visual understanding: the student needs to conceptually understand how a given visual representation denotes information and has to be able to distinguish conceptually relevant from irrelevant visual features. In sum, ***learning principle 3*** suggests that instruction should engage students in explicit sense-making processes aimed at explaining connections among multiple visual representations.

Empirical research in a variety of domains has yielded a number of ***design guidelines*** for instruction that supports connectional understanding (guideline 3.1). First, students should *verbally explain* which features of the various representations depict corresponding concepts (i.e., similarities between the representations) and which features show complementary information (e.g., number lines can more easily depict fractions larger than 1, compared to circles and rectangles). For example, prompting students to such self-explain mappings has been shown to enhance learning of domain knowledge (Berthold and Renkl 2009; van der Meij and de Jong 2011).

Second, as with visual understanding, students should *actively establish* mappings between visual features of the representations (guideline 3.2).

Third, because students tend to focus on surface features instead of conceptually relevant features (Ainsworth et al. 2002; Rau et al. 2014), they need *assistance* in identifying relevant perceptual features (guideline 3.3). Such assistance can be provided by giving feedback on student-generated connections (Bodemer and Faust 2006; van der Meij and de Jong 2006) or color coding (Berthold and Renkl 2009). Assistance is particularly important for students with low prior knowledge (Bodemer and Faust 2006; Stern et al. 2003) and for complex problems (van der Meij and de Jong 2006).

Figure 5 shows an example of instructional support for connectional understanding from the Fractions Tutor. First, students are prompted to *explain* these mappings (guideline 3.1). To this end, students receive self-explanation prompts at end of each problem (section C in tutor screen shown in Fig. 5). The prompts ask students to relate the two representations by reasoning about how they depict fractions. Second, to help students *actively establish* mappings between visual representations (guideline 3.2), the Fractions Tutor uses worked examples (Renkl 2005). Students

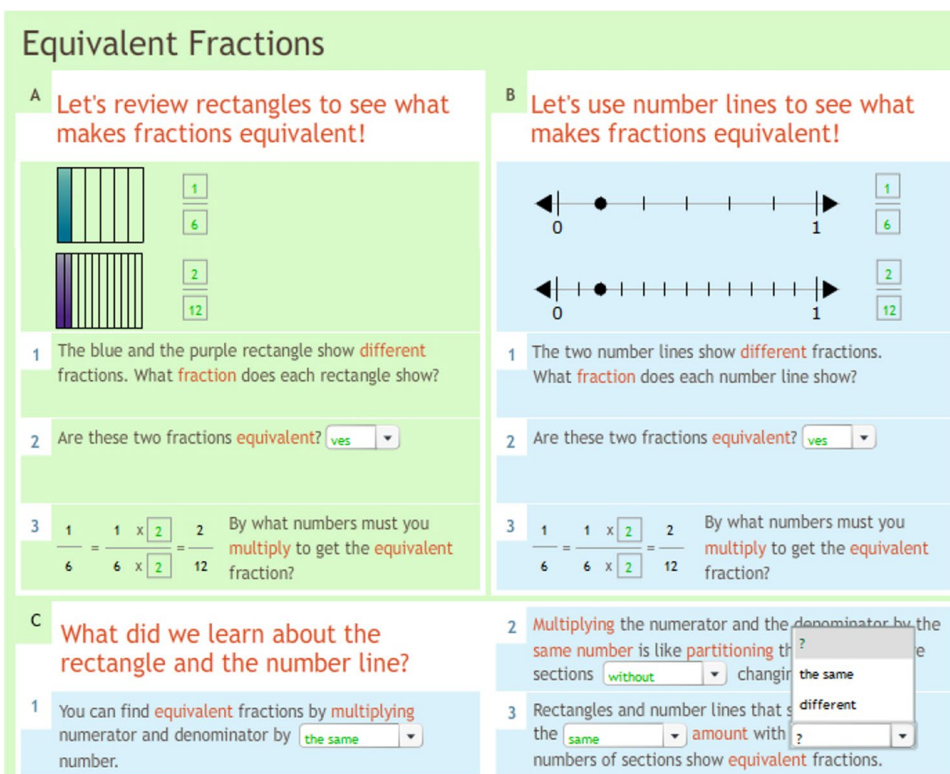


Fig. 5 Example activity from the Fractions Tutor to support connectional understanding

are first presented with a worked example (Fig. 5a) that uses a visual representation they are likely more familiar with to demonstrate how to solve a fractions problem (e.g., a rectangle). Students complete the last step of the worked-example problem (step A.3). With the worked example still on the screen, they then see an analogous problem requiring them to use the number line (Fig. 5b). Students are prompted to use the rectangle to help them complete the number line activity, so as to encourage them to establish mappings between corresponding visual features (e.g., the sections between 0 and the dot in the number line correspond to the shaded sections in the rectangle because both visual features show the numerator). Finally, students receive *assistance* (guideline 3.3) in the form of feedback and on-demand hints on problem-solving activities and on their responses to self-explanation prompts.

An experiment with the Fractions Tutor evaluated the effectiveness of providing support for connectional understanding (Rau et al. 2016). 4th- and 5th -grade students ($N=428$) were randomly assigned to work with one of several versions of the tutor that either did or did not contain the support for connectional understanding just described. Results showed that a version of the Fractions Tutor that did include support for connectional understanding yielded higher learning outcomes on a fractions knowledge test than versions that did not include support for connectional understanding.

In sum, instructors can implement learning principle 3 by asking students to actively identify visual features in different visual representations that show corresponding concepts, and by asking them to explain differences between visual representations, as one may show information the other one does not. It is likely that students need assistance in explaining these mappings, especially if the domain is complex. In this case, instructors can provide worked examples, but they should make sure students still become active in explaining the connections in their own words, for instance by prompting students to self-explain the mappings shown in the worked examples.

4.4 Supporting connectional fluency: expose students to many opportunities to translate among visual representations

As with visual fluency, students acquire connectional fluency by engaging in implicit, non-verbal inductive learning processes. Hence, similar to support for visual fluency, support for connectional fluency should provide students with experience translating among many example representations without explicit instruction (Koedinger et al. 2012). Hence, ***learning principle 4*** suggests that instruction should engage students in implicit fluency-building processes aimed at translating among visual representations.

Research on perceptual learning (Kellman and Massey 2013) in a variety of domains provides ***design guidelines*** for instruction that supports connectional fluency. First, students should gain experience in *discriminating and categorizing* multiple visual representations (guideline 4.1). Second, students should receive *immediate feedback* on these discrimination and classification activities (guideline 4.2). Third, students should practice with *many varied example representations*, sequenced such that consecutive examples emphasize relevant visual features (guideline 4.3).

Figure 6 shows an example from the Fractions Tutor that illustrates these guidelines. First, students are asked to *discriminate and categorize* visual representations. In Fig. 6, they have to sort the representations provided in the box on the left into sets of equivalent fractions by dragging them into the boxes on the right. Second, students receive *immediate feedback* on this task (e.g., a green halo indicates the representation was dropped in the correct box). Third, students receive practice opportunities with *many varied visual representations*. To encourage students' engagement in non-verbal processes, students are instructed to solve the problem visually by estimating the relative size of the fractions, rather than by solving it conceptually or computationally.

The previously mentioned experiment with the Fractions Tutor also evaluated whether providing support for connectional understanding is effective (Rau et al. 2016). A second factor (i.e., besides support for connectional understanding) was whether or not students received the support for connectional fluency just described. Results showed that support for connectional fluency was effective, but only when combined with support for connectional understanding.

In sum, instruction can implement learning principle 4 by exposing students to a variety of examples of the visual representations and asking students to match representations not by verbally explaining the connections but by relying on visual features. Many instructional materials engage implicit fluency-building processes; for instance, there are games in which students compete with one another as they try to find matching representations depicted on a screen or on cards. Given the current state of research, it seems important that instructors ensure that students are exposed to systematic variations of visual features and that they receive immediate feedback on whether their connections are correct or incorrect. Further, support for connectional fluency should be combined with support for connectional understanding.

5 Open questions and future directions

Research that uses rigorous psychological methods to investigate how best to help students acquire representational

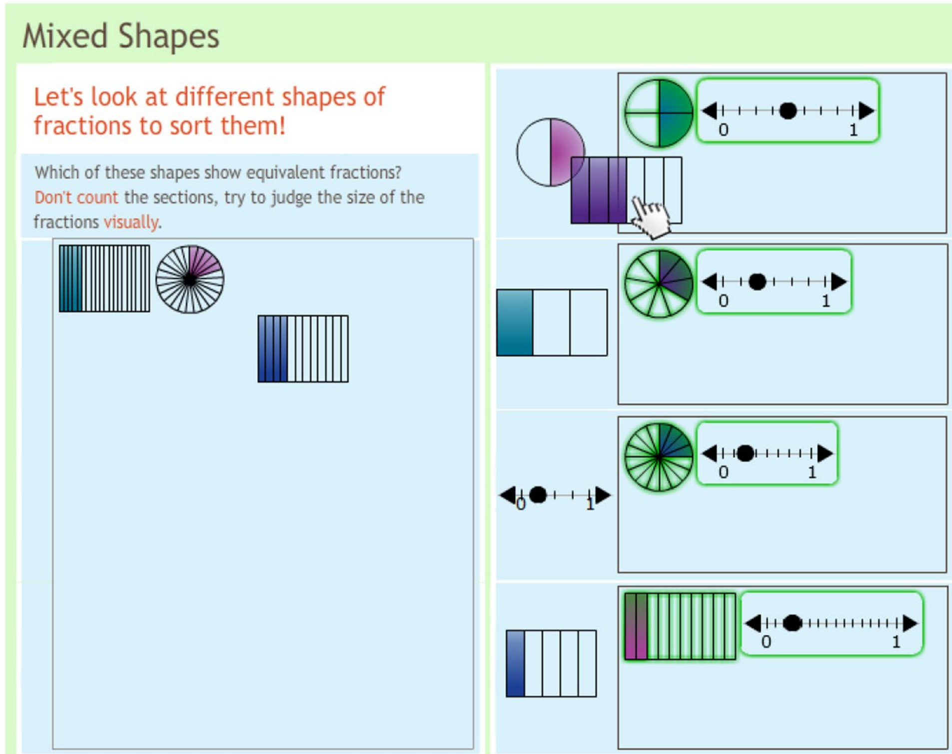


Fig. 6 Example activity from the Fractions Tutor to support connectional fluency

competencies involved in learning mathematics with multiple visual representations is still relatively novel. Consequently, there are many open questions that future research should address—for the specific domain of fractions and for mathematics learning more generally. Here we discuss three that we find particularly interesting.

First, we know relatively little about how the different representational competencies described above build on one another. For example, when students acquire connectional understanding, is it helpful if they have some level of visual understanding with at least one of the visual representations? This is an important question because it may suggest alternatives to relying on text-based representations to help students acquire visual understanding: students may be able to use a well-understood or privileged visual representation to understand a novel visual representation and acquire, at the same time, connectional understanding *and* visual understanding of the novel representation (see Acevedo Nistal et al. 2009).

Second, our focus has primarily been on how best to use multiple visual representations without addressing the question of which representations to choose. A second, related question concerns the possibility that some visual representations may be intuitively more accessible than others because they align with the structure of human cognitive architecture. For instance, Tversky et al. (2000; also see Tversky 2011) hypothesized that some visual representations convey meaning more intuitively than others because the human brain seems to be sensitive to perceiving their referents from their physical structures. One obvious example is spatial iconicity, where graphic space is used to represent scaled versions of real space. To the extent that privileged representations can be found in specific domains, employing them could help solve the representation dilemma (but see Acevedo Nistal et al. 2009 on subjective factors). Specifically, deploying them as anchor representations might help optimize the web of meaning that emerges from use multiple representations.

Privileged representations seem to exist for at least some subconstructs of the fractions/rational number megaconcept. Research suggests that many mammalian species possess a dedicated system that can process the magnitudes of nonsymbolic fraction analogs (see Jacob et al. 2012 for a review). Evidence for these accounts continue to mount, as sensitivity to nonsymbolic ratio magnitudes has been found by a number of experiments involving a range of different methods and subjects (see Fig. 7), including non-human primates (e.g.,Vallentin and Nieder 2008), preverbal infants (e.g., McCrink and Wynn 2007), elementary school-age children (e.g., Huttenlocher et al. 2002), and adults (Matthews and Lewis 2017). Indeed, some research suggests that these nonsymbolic fraction magnitudes are processed automatically (Fabbri et al. 2012; Jacob and Nieder 2009; Matthews and Lewis 2017; Yang et al. 2015).

Indeed, research with young children suggests perceptual sensitivity to nonsymbolic fraction analogs is

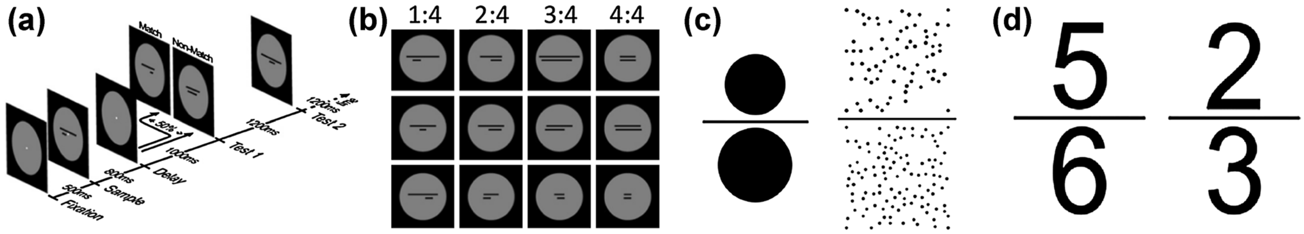


Fig. 7 Tasks used to investigate the cognitive primitives account. Vallentin and Nieder (2008) had monkeys and humans complete a forced-choice ratio matching task (**a**). Participants saw several distinct yet equivalent instantiations of each ratio (**b**). Matthews and Chesney (2015) showed that adult humans could accurately complete cross format nonsymbolic ratio matching tasks (**c**) faster than they completed symbolic ratio matching tasks (**d**)

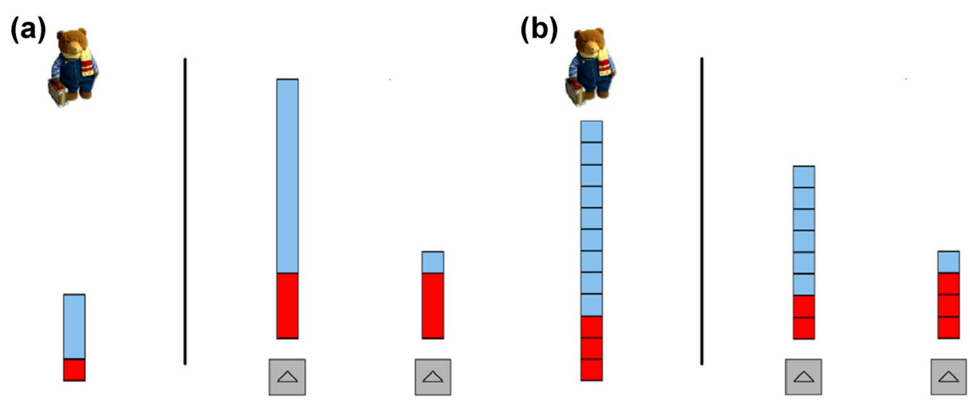
preferentially leveraged by a particular class of nonsymbolic representations, specifically those composed of unpartitioned, continuous components. For instance, Boyer et al. (2008) compared two versions of a proportion matching task with K-4th grade students. Given a target proportion (Fig. 8, left) students had to select one of two choices (Fig. 8, right). They performed significantly worse when ratios were partitioned into countable pieces, which encouraged counting (Fig. 8a) than when they were unpartitioned (Fig. 8b). This finding suggests that young children possess the necessary perceptual abilities for rudimentary proportional reasoning but that, partitioned representations are less effective than continuous representations for supporting this reasoning (see also Boyer and Levine 2012; Jeong et al. 2007). This body of findings led Lewis et al. (2015) to propose that early fractions instruction should incorporate unpartitioned continuous nonsymbolic ratios due to their potential standing as privileged visual representations.

We do not seek to endorse this recommendation; rather, we feel a brief review of this argument illustrates why research should investigate the possibility that there may be privileged visual representations when choosing from vast spaces of available visual representations. If there indeed are privileged representations, they may enhance visual understanding and perhaps serve as anchors for cultivating connectional understanding, and these in turn may enhance further representational competencies. The suggestion that some visual representations may be inherently better-suited than others for communicating their referents in particular domains is tantalizing, but there is a dearth of research in this area, and the question remains an open one.

Finally, we know relatively little about how specific classroom-based implementations of the design principles described above affect students’ learning. Many interventions can help students explain how a given visual representation of fractions depicts information (design principle 1), in ways that conform to the guidelines described above. For example, an instructor may have students collaboratively discuss representational conventions and provide assistance in the form of verbal feedback. Alternatively, an instructor may provide worked examples on worksheets that ask individual students to identify visual features that show particular concepts and provide assistance in the form of written feedback. Which implementation is more effective and why?

To address these open questions, the research community would benefit from reciprocal collaborations among mathematics education researchers, cognitive scientists, and research-oriented instructional practitioners. Such collaborations would promote rigorous experiments that can evaluate the effectiveness of realistic interventions designed to support representational competencies in real classrooms

Fig. 8 Partitioned representation (**a**) versus continuous representation (**b**) used by Boyer and Levine (2012)



over longer periods. For example, our understanding of how representational competencies allow students to navigate the representation dilemma would greatly benefit from gathering data about students' visual understanding and fluency and about their connectional understanding and fluency over the course of an entire curriculum.

6 Conclusions

Visual representations are powerful educational tools that are commonly used in mathematics instruction because they have the potential to significantly enhance students' mathematics learning. However, use of multiple visual representations can also impede students' learning; for example, if students fail to understand how they depict information or if they fail to make connections among multiple visual representations. Hurdles to using visual representations result from the representation dilemma: Students often use unfamiliar visual representations to learn about unfamiliar concepts. We provided an overview of representational competencies that allow students to navigate the representation dilemma, using fractions as an illustrative domain. We presented four learning principles and accompanying design guidelines that can inform instructional interventions that support students in acquiring these representational competencies. While the principles we described are based on research in a variety of domains and likely broadly applicable, we illustrate possible implementations of these principles and provide evidence for their effectiveness in the domain of fractions. Finally, we described several open questions that remain to be addressed. Because representational competences play a key role in students' mathematics learning, research that yields principles for the design of supports for representational competencies is likely to yield more successful mathematics instruction.

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