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Teachers' and Researchers' Beliefs About the Development of Algebraic Reasoning

Mitchell J. Nathan, *University of Colorado*
Kenneth R. Koedinger, *Carnegie Mellon University*

Mathematics teachers and educational researchers ordered arithmetic and algebra problems according to their predicted problem-solving difficulty for students. Predictions deviated systematically from algebra students' performances but closely matched a view implicit in textbooks. Analysis of students' problem-solving strategies indicates specific ways that students' algebraic reasoning differs from that predicted by most teachers and researchers in the sample and portrayed in common textbooks. The Symbol Precedence Model of development of algebraic reasoning, in which symbolic problem solving precedes verbal problem solving and arithmetic skills strictly precede algebraic skills, was contrasted with the Verbal Precedence Model of development, which provided a better quantitative fit of students' performance data. Implications of the findings for student and teacher cognition and for algebra instruction are discussed.

Key Words: Algebra; Children's strategies; Cognitive theory; High school, 9–12; Prealgebra; Problem solving; Secondary mathematics; Teacher beliefs

In our current investigation we explore the relationship between teachers' and researchers' predictions about the development of algebraic reasoning and students' performances. Teachers' beliefs about students' ability and learning greatly influence their instructional practices. The study of these beliefs has revealed that teachers generally report that information about students is the most important factor in their instructional planning (Borko & Shavelson, 1990), and teachers consider students' ability to be the characteristic that has the greatest influence on their planning decisions. When students shift from an arithmetic approach to problem solving to an algebraic approach, teachers must also shift their practices and their views of the learner. Any improvement in our understanding of teachers' views of the development of students' knowledge and problem-solving abilities strengthens our picture of the complexities of teaching and may ultimately enhance programs for teacher preparation. We extend work on previous models both of teachers'

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cognition and of the ways professional practitioners' knowledge and beliefs shape their instructional practices (Borko & Livingston, 1989; Knuth, 1999; Nathan, Knuth, & Elliott, 1998; Schoenfeld, 1998; Shulman, 1986; Thompson, 1992).

In this article we report on empirical results concerning high school teachers' and mathematics educational researchers' beliefs about the factors that make algebra problems difficult for beginning algebra students. We include the responses of educational researchers because of the central role they play in teacher education, professional development, and the development of instructional materials and activities. By examining the views of researchers as well as teachers, we hope to understand the extent to which certain beliefs about algebraic problem solving and development are held.

In this study we made comparisons between teachers' and researchers' predictions of the relative difficulties of a set of theoretically designed problems and students' problem-solving performances (Koedinger & Nathan, 1998; Koedinger, Nathan, & Tabachneck, 1996). In the analyses that follow, we examine discrepancies between teachers' and researchers' predictions and students' performances. The prediction data were also compared to the sequencing of problems presented in some prealgebra and algebra textbooks. Results from the analyses of students' problem-solving strategies lead us to suggest specific ways that students' algebraic reasoning differs from the views of development of algebraic reasoning commonly held by teachers and researchers and the views presented in popular algebra textbooks.

Algebra has several interleaving aspects. First, algebra can be seen as generalized arithmetic, including the use of literal symbols such as letters as references to unknown quantities and the generalization of arithmetic operations as they apply to letters (Kieran, 1992; MacLane & Birkhoff, 1967; Usiskin, 1988, 1997). Second, algebra refers to the use of formal mathematical structures to represent relations and includes the procedures that operate on those structures (Kieran, 1992; Usiskin, 1988). Third, algebra can be defined as a formal means to describe the relationships among quantities (Usiskin, 1988, 1997). We acknowledge the extent of the domain of algebra. In this study we limit our scope and specifically address the aspects of algebra and arithmetic as they relate to equation and word-problem solving.

THEORETICAL FRAMEWORK: FACTORS AFFECTING PROBLEM-SOLVING DIFFICULTY

We first review research in which the relative effects of certain factors on algebra and arithmetic problem-solving difficulty were considered. We specifically considered two important factors: (a) the position of the unknown quantity in the problem and (b) the linguistic presentation of the problem. The body of work on arithmetic story-problem solving of younger children (e.g., Carpenter, Fennema, & Franke, 1994; De Corte, Greer, & Verschaffel, 1996; Riley, Greeno, & Heller, 1983) provides firm methodological and theoretical bases for the study of high school students' algebraic reasoning and its development and impediments.

Riley et al. (1983), for example, found that problem difficulty is strongly affected by the role (or position) of the unknown quantity within the problem statement. Consider the case of a *result-unknown problem,* in which the unknown quantity is the result of the events or mathematical operations described in the problem. An example result-unknown problem is shown in Problem 6 of Table 1, with the symbol x standing for the unknown quantity. To find the value of x in this example, the problem solver applies the indicated arithmetic operations on the left-hand side of the equation, first subtracting, then dividing. Problem 5 and Problem 4 (Table 1) are also result-unknown problems, presented in two different verbal formats. Because result-unknown problems can be solved through the direct application of arithmetic operations, these can be considered arithmetic-level problems.

In *start-unknown problems,* the unknown value (such as the hourly wage in Problem 1 of Table 1 or x in Problem 3) refers to a quantity needed to specify a relationship (Carpenter et al., 1994; Riley et al., 1983). Start-unknown problems tend to subvert simple modeling and direct-calculation approaches of arithmetic problems and often require algebraic methods or more sophisticated modeling (Hall, Kibler, Wenger, & Truxaw, 1989). Because they can be solved through the application of standard algebraic procedures, start-unknown problems can be considered algebra-level problems.

Table 1
Sample Problems Used to Elicit Difficulty-Ranking Judgments From High School Teachers and Mathematics Education Researchers

Problem type	Problem statements
Problem 1 Start unknown; story problem	When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. Then he added the $66 he made in tips and found he earned $81.90. How much per hour did Ted make?
Problem 2 Start unknown; word equation	Starting with some number, if I multiply it by 6 and then add 66, I get 81.90. What did I start with?
Problem 3 Start unknown; symbolic equation	Solve for x: $x \cdot 6 + 66 = 81.90$
Problem 4 Result unknown; story problem	When Ted got home from his waiter job, he took the $81.90 he earned that day and subtracted the $66 he received in tips. Then he divided the remaining money by the 6 hours he worked and found his hourly wage. How much per hour does Ted make?
Problem 5 Result unknown; word equation	Starting with 81.90, if I subtract 66 and then divide by 6, I get a number. What is it?
Problem 6 Result unknown; symbolic equation	Solve for x: $(81.90 – 66) / 6 = x$

Note. Result-unknown problems are considered to be arithmetic; start-unknown problems are considered to be algebraic.

Riley et al. (1983) examined the performance of first-grade students solving simple one-step problems with whole numbers. They found that although students correctly solved 100% of result-unknown (arithmetic) problems, they correctly solved only 33% of the start-unknown (algebra) problems. We need not suspect that these students were using algebraic methods to solve these problems. Use of number-fact retrieval and counting strategies can explain the nontrivial performance on start-unknown and open-sentence problems at this level (Briars & Larkin, 1984; Kieran, 1992). This general pattern of problem-solving performance favoring result-unknown over start-unknown problems has also been found in mathematical problem solving at the college level for multistep problems with rational numbers (Koedinger & Tabachneck, 1995).

Investigators have examined the performance differences between problems presented in symbolic (or computational) formats and those presented in linguistic or verbal formats, such as word and story problems. Story problems (e.g., Problems 1 and 4 of Table 1) are those presented in a verbal format with contextual information about the problem situation that can be used by the solver as a source of problem elaboration, reframing, and solution constraints (cf. Baranes, Perry, & Stigler, 1989). Symbolic-equation problems (Table 1, Problems 3 and 6) are typically described as number sentences.

There is a presentation format intermediate to the story-problem and symbolic-equation formats. The *word-equation* format (Table 1, Problems 2 and 5) verbally describes the relationship among pure quantities (both known and unknown) with no story context. A common example of this type of problem is the “pick a number” game (Usiskin, 1997).

Although the greater ease of result-unknown problems compared with start-unknown problems is widely agreed upon, findings in the literature are inconsistent in examinations of the relative difficulties of verbal problems versus symbolic problems. Researchers have found circumstances in which computational problems are solved more readily than word problems (e.g., Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980). Researchers have also found performance advantages for problems with contexts compared with symbolic equations (e.g., Carraher, Carraher, & Schliemann, 1985; Carraher & Schliemann, 1985). Brazilian children who regularly engaged in street trade, for example, solved arithmetic problems more readily when the problems were presented in a practical context such as a story, an action sequence, or, preferably, as a real-life interaction of the street markets (Carraher et al., 1985; Guberman, 1987; Saxe, 1988). Specifically, contextualized problems presented either as typical word problems or as problems situated in a commercial transaction led to greater levels of performance than symbolically presented problems.

To add to the picture, Baranes et al. (1989) found that Brazilian children, but not U.S. children, exhibited this performance advantage for problem context. Strategy selection appeared to be an important mediating variable. Decontextualized problems tended to elicit less successful mental strategies in Brazilian children, with no corresponding differences among the sample of U.S. students. However, the U.S.

students did show an advantage for problems in which the context (e.g., money or time) closely matched the numbers used (e.g., multiples of 25 cents or 15 min), suggesting that the activation of real-world knowledge facilitated problem-solving performance (cf. Nathan, Kintsch, & Young, 1992).

To further understand the relative effects of context and other factors on problem-solving difficulty, we (Koedinger & Nathan, 1998; Koedinger et al., 1996) studied high school and college students in several factorially designed assessments. We termed this investigation *difficulty-factors analysis* (DFA) because we sought to systematically examine the factors affecting students' problem-solving difficulties. The findings from the 1998 study are reviewed here in some detail because of their significance in interpreting the current study of teachers' and researchers' beliefs.

In our previous work (Koedinger & Nathan, 1998), students solved problems based on six problem types that resulted from varying items along three forms of presentation formats (verbal stories with context, word equations with no context, and symbolic equations) and two placements of the unknown quantity (result-unknown or start-unknown problems) as shown in Table 1. Two assessments of students' achievement on different problem types were undertaken in an urban high school. Both took place close to the end of the academic year. The students were enrolled either in Algebra I or in Geometry (in which case they had completed Algebra I a year earlier). The courses in Algebra I followed a standard curriculum. There were 76 students tested in the first assessment (Table 2) and 171 in the replication assessment (Table 3) one year later.

Unknown Values

Students in the original study (n = 76) exhibited much lower performance levels on start-unknown (algebra) problems than on result-unknown problems (see the rows of Table 2). Students correctly solved 50% of the start-unknown problems, and they correctly solved 64% of the result-unknown (arithmetic) problems, leading to a significant effect of unknown position on problem difficulty, $F(1, 75) = 48.9$, $p < .0001$.

The following year, students in the replication study (n = 171) showed a similar pattern of results (Table 3). They correctly solved 46% of the start-unknown prob-

Table 2
*Student Performance in Percentages (*n *= 76) on an Assessment of Problems From the Six Problem Types of Table 1*

	Verbal		Symbolic	
Value unknown	Story	Word equation	Symbolic equation	Total
Result unknown (arithmetic)	73	67	53	64
Start unknown (algebra)	59	54	37	50
Total	66	61	45	57

Note. Data are taken from Koedinger and Nathan (1998).

lems whereas 70% of the result-unknown problems were solved. This result produced a significant effect of unknown position on problem difficulty, $F(1, 170) = 138$, $p < .0001$.

Table 3
*Replication of Student Performance in Percentages (*n *= 171)*
on an Assessment of Problems From the Six Problem Types of Table 1

	Verbal		Symbolic	
Value unknown	Story	Word equation	Symbolic equation	Total
Result unknown (arithmetic)	80	74	56	70
Start unknown (algebra)	60	48	29	46
Total	70	61	43	58

Note. Data are taken from Koedinger and Nathan (1998).

Presentation Formats

As discussed above, the format in which a mathematical problem is presented also bears on problem difficulty (Baranes et al., 1989; Carpenter et al., 1980; Carraher, Carraher, & Schliemann, 1987). We found that high school students in the original sample ($n = 76$) experienced a nearly 20% drop in performance when solving symbolic equations compared with matched verbally presented problems with or without a context (Table 2). These differences resulted in a significant effect for presentation format, $F(2, 75) = 12.6$, $p < .0001$. A post hoc test ($p < .01$) revealed that symbolic-equation problems were significantly less likely to be correctly solved than either story problems or word equations, whereas algebra story problems and algebra word-equation problems were found to be equal in difficulty (Table 2).

Although no interaction between position of unknown and presentation format was evident, a post hoc analysis ($p < .01$) revealed that algebra story problems and algebra word problems were equal in difficulty to arithmetic symbol problems. This finding suggested two possible causes that could have been operating simultaneously. First, there could have been an inhibiting effect of the symbolic format that burdens the students' cognitive resources for arithmetic reasoning on the arithmetic equations. Second, there could have been a facilitating effect of verbal format (regardless of the presence of problem *context*) that lessened the demands of the start-unknown structure found in algebra story and word-equation problems.

Students in the replication study ($n = 171$) also showed the facilitating effects of verbal-presentation format on problem-solving performance. Solution success rates for symbolic equations were 25% less than for story problems and nearly 20% less than for word equations (see Table 3). As in the original study, these differences led to a significant effect for presentation format, $F(2, 170) = 38.4$, $p < .0001$. A post hoc test revealed that symbolic-equation problems were significantly more difficult than either story problems or word-equation problems, $p < .01$. However, the replication data differed from the results of the original study. A post hoc test ($p < .01$) revealed that result-unknown word equations (e.g., Problem 5) were significantly more difficult to solve than result-unknown story problems (Problem 4) for these students.

Three generalizations can be drawn from the high school student data: (a) Start-unknown (algebra) problems are harder for these students than result-unknown (arithmetic) problems ($p < .001$); (b) symbolic-equation problems are harder than both word-equation problems and story problems ($p < .001$); and (c) verbal algebra problems are equal in difficulty to symbolic arithmetic problems.

Students' Solution Strategies

We observed four major types of solution strategies used by the high school students to solve the six classes of problems discussed earlier. The first two strategies—arithmetic and algebraic methods—are the standard school-taught methods. The other two strategies—guess-and-test and unwinding—are informally adopted and invented strategies.

The *guess-and-test* strategy refers to the class of model-based methods used for iterative analysis or "hand simulations" of the events of the problem (e.g., Hall et al., 1989; Kieran, 1992). In the example of the application of this strategy (see Figure 1), the student was solving a word-equation start-unknown (algebra) problem by guessing possible numbers that could be added to the number 25 to yield a sum of 66.40. Because the student never achieved this goal, the search for the original number (i.e., some number multiplied by 4) was apparently neglected. In such two-step problems, steps sometimes get dropped because of the large demands that problem-solving activities place on one's limited working memory (Anderson, Reder, & Lebiere, 1996).

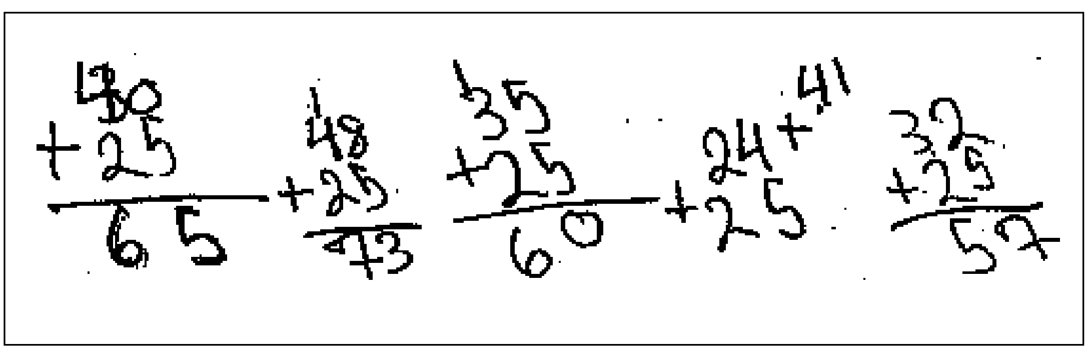

Figure 1. Guess-and-test strategy used by Student #103.

In contrast to the guess-and-test method, the *unwinding* method allows the student to "work backwards" from the givens of the problem and "unwind" or undo the imposed quantitative constraints in order to isolate the unknown (cf. Graves &

Zack, 1996; Kieran, 1992; Kieran & Chalouh, 1990; Polya, 1957). Unwinding often inverts the steps referred to in a story problem or the order of mathematical constraints provided in a symbolic- or word-equation problem. Figure 2 shows work a student did while solving a start-unknown story problem (Problem 1, Table 1) using the unwinding method. The student unwinds the waiter's tips from the total earnings of $90 and obtains a value of 24. The student then unwinds the multiplication by 6 (for the 6 hours that Ted worked) using long division to obtain the initial quantity of $4.00—the waiter's hourly wage.

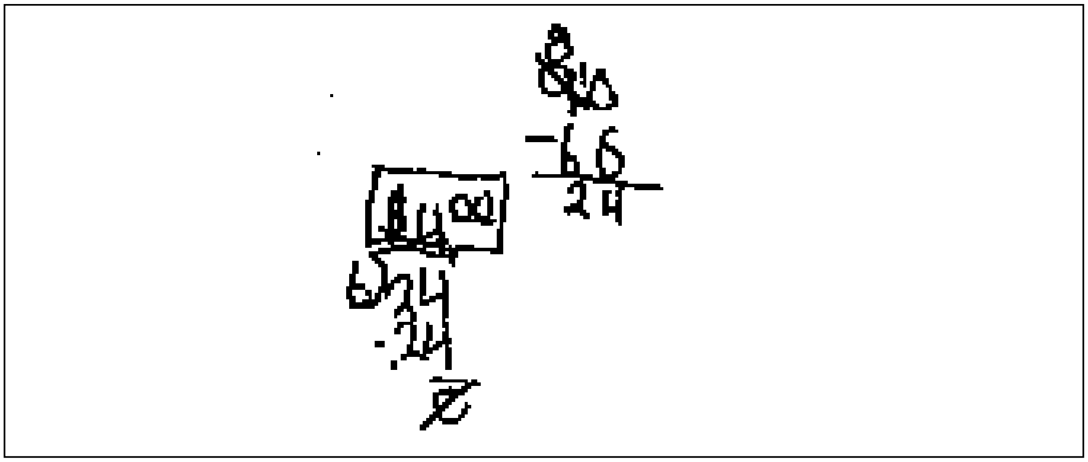

Figure 2. Unwinding strategy used by Student #99.

Unlike using traditional algebraic approaches such as transposing terms while maintaining a balanced equation, employing the unwinding strategy circumvents use of equations or symbolic placeholders for unknown quantities. The student operates directly on the numbers in a computational way rather than operating on the symbol structure of the equation (Kieran, 1992). Unwinding may be done verbally by the solver or through the solver's written work (Koedinger & Tabachneck, 1995). By unwinding the mathematical relationships step-by-step, the solver systematically does two things. First, he or she transforms a multistep problem into a set of simpler one-step problems, each of which is solved separately by unwinding a single operation. Second, the solver applies arithmetic in each step in the unwinding procedure, using the inverse operations of those given in the original problem. Using this approach, one essentially circumvents the need to perform (or even know about) the rules of algebraic symbol manipulation or equation balancing. We hypothesize that students' problem-solving performances on algebra story problems and word-equation problems are at the same level as on solving arithmetic symbolic-equation problems (a finding reported above) because the unwinding strategy essentially transforms multistep algebra problems into a sequence of single-step arithmetic problems.

An analysis of problem-solving-strategy selection from the initial student data (n = 76) was conducted. Not surprisingly, the arithmetic strategy was used overwhelmingly in solving result-unknown problems. For the purposes of this article, we focus on solution methods used to solve start-unknown (algebra) problems. Even though all students in the sample had extensive instruction in algebraic methods, verbal problems seldom elicited symbol-manipulation methods. More than half of the algebra story problems elicited the unwinding strategy (Figure 2) from students, and more than half of the algebra word-equations elicited either unwinding or a guess-and-test approach (Figure 1). In contrast, an alarming 30% of symbolic equations—more than twice the rate for all other problem types—elicited no response from these algebra students. When students did respond to algebra equations, they tended to stay within the algebraic formalism and apply symbol-manipulation methods or to opt for the iterative guess-and-test method. Algebraic symbolic-equation problems elicited alternative methods about half as often as algebra story problems and word-equation problems.

This pattern of strategy selection is notable because different solution strategies had different success rates. The unwinding and guess-and-test methods had the highest likelihood of success (around 70%) and led to the greatest number of correct solutions when they were applied to start-unknown problems. Symbol-manipulation strategies were effective about half the time. One reason offered for this performance advantage is that the informal methods we observed employ built-in validity checks on the answers (Tabachneck, Koedinger, & Nathan, 1994).

A variety of students' solution methods have been noted in the literature on algebra-level problem solving (e.g., Hall et al., 1989; Kieran, 1988, 1992; Petitto, 1979; Polya, 1957; Tabachneck et al., 1994). However, as Rachlin (1989) noted, "There is a need for research on the learning and teaching of the curriculum at two levels—that of the students and that of the teachers" (p. 259). Because teachers' beliefs about student ability and learning greatly influence their instructional practices, we set out to observe the effect of problem features (unknown position and presentation format) on high school teachers' predictions about students' difficulties in solving the problems.

Mathematics educational researchers' views also play a vital role in shaping classroom practices and in designing curricula. If teachers' misconceptions about algebra problem solving are not shared by researchers in the field, teachers would be likely to relinquish these views in light of the findings available to researchers. However, if these views are also held by researchers who are informed by the current research, we may expect that these views are firmly rooted and may be more resistant to change. In this case researchers would need to be alerted to these widely held misconceptions so they could consider new approaches for improving teachers' understanding of students' thinking. For these reasons we include in our investigation the judgments of algebra researchers as well as those of teachers.

METHOD

Participants

Mathematics teachers (n = 67) and mathematics educational researchers (n = 35) participated in the study. The teachers were from the same southeastern state and taught in a wide range of settings, including predominantly minority-based inner city schools, rural communities, and middle-income suburban areas. They were recruited from a teacher workshop during the summer of 1995, and their teaching responsibilities included 7th through 12th grades. The researchers were dispersed throughout the United States and were recruited via an Internet discussion group specifically focused on issues of algebraic thinking and instruction.

Procedure

The teacher participants were asked to individually rank order 12 mathematics problems from easiest to most difficult. The 12 problems comprised 2 problems for each of the six types discussed earlier (see Table 1). These problems were chosen because they were representative of problems in the mathematics textbooks used in the teachers' school districts.

The teachers were given the following instructions:

> Below are 12 problems that are representative of a broader set of problems that are typically given to public school mathematics students. My colleagues and I would like you to help us by answering this brief (10 min) survey. We are happy to share the results we obtain with you this spring.
>
> What we would like you to do:
> Rank these 12 problems starting with the ones you think are easiest for your students to the ones you think are hardest. You can have ties if you like.

Unlike the teachers, those researchers who participated in this study were self-selected on the basis of their own interest. All were members of an online discussion group on algebra learning and instruction. Researchers received six of the problems given to the teachers, one from each of the six problem types. They were told, "Below are problems that are representative of a broader set of problems that have been given to urban high school students at the end of an Algebra I course—they are mostly 9th-grade students." Participant researchers were then asked to "rank these problems starting with the ones you think were more difficult for these students to the ones you think were easier." The materials were distributed and collected over e-mail.

The problems from the survey were generated using the three presentation formats—story, word equation, and symbolic equation—and the two unknown positions—result and start. Note that the underlying mathematical relationships are the same across the six problem types given to participants. Neither this underlying structure nor the problem variations were discussed with the participants, but they were important in our subsequent analyses of participants' predictions and students' performances.

RESULTS

The average rank-orderings produced by the mathematics teachers and the researchers in our sample are presented, respectively, in the first and second rows of Table 4. Each of the rank orderings was analyzed using a 2-way, repeated measures ANOVA with position of the unknown (result vs. start) and presentation format (story vs. word equation vs. symbol) as within-subjects factors and difficulty rank as the dependent measure. The divisions reflect statistically significant differences ($p < .05$) among the rank-ordered data.

Table 4
Difficulty Ranks Given by Teachers and Researchers and Determined by Students' Performances

	Difficulty level			
Data set	Easy[a]	Medium-easy[a]	Medium-hard[a]	Hard[a]
Teachers' mean rank ($n = 67$)	RU-Eqn (P6) RU-Word (P5) RU-Story (P4)	SU-Eqn (P3)	SU-Story (P1)	SU-Word (P2)
Researchers' mean rank ($n = 35$)	RU-Word (P5) RU-Story (P4) RU-Eqn (P6)	SU-Eqn (P3)		SU-Word (P2) SU-Story (P1)
Student performance in original study ($n = 76$) [and % correct]	RU-Story (P4) [73%] RU-Word (P5) [67%]	SU-Story (P1) [59%] SU-Word (P2) [54%] RU-Eqn (P6) [53%]		SU-Eqn (P3) [37%]
Student performance in replication study ($n = 171$) [and % correct]	RU-Story (P4) [80%] RU-Word (P5) [74%]	SU-Story (P1) [60%] RU-Eqn (P6) [56%] SU-Word (P2) [48%]		SU-Eqn (P3) [29%]
Symbol Precedence Model	P6 P5 P4	P3		P2 P1

Note. RU stands for result-unknown problems. SU stands for start-unknown problems.
[a]Difficulty divisions (Easy, Medium-easy, Medium-hard, Hard) indicate significant differences ($p < .05$) in mean ranking by teachers and researchers or in students' performance levels.

Unknown Values

Teacher data. Across all problem types, 84% of the 67 teachers ranked result-unknown (arithmetic) problems (Problems 4, 5, and 6 of Table 1) as significantly easier than start-unknown problems. Arithmetic symbolic equations (Problem 6) were predicted by teachers to be significantly easier to solve than arithmetic word equations (Problem 5), whereas word equations were expected to be slightly easier to solve than arithmetic story problems (Problem 4). These ranking data showed a significant main effect for the unknown-value factor, $F(1, 134) = 5.9$, $p < .02$. Six percent of the teachers viewed start-unknown problems as easier than result-unknown problems, and 10% were inconsistent in their rankings.

Researcher data. The mathematics researchers ($n = 35$) showed strong agreement with teachers. About 66% of the respondents ranked start-unknown problems as consistently harder than result-unknown problems across the three presentation forms, and 34% ranked result-unknown problems as more difficult in some but not all cases.

Relation to student performance. The relative ranks of problem difficulty for students in both samples are shown in Table 4. As predicted by teachers and researchers, students exhibited lower performance levels on start-unknown (algebra) problems than on result-unknown problems. As Tables 2 and 3 show, the high school students in our two samples scored between 14% and 24% higher on result-unknown than on start-unknown problems.

Presentation Format

Teacher data. After we collapsed results across the unknown-value factor, the data showed that 42% of teachers ranked symbolic equations (e.g., Problems 3 and 6) as consistently easier to solve than word equations on average, and 49% ranked symbolic-equation problems as consistently easier to solve than story problems (e.g., Problems 1 and 4) on average. Fewer than 30% ranked verbal algebra problems (story and word-equation problems combined) as being as easy to solve as symbolically presented arithmetic problems. Only 8 teachers (12%) consistently ranked symbolic equations as harder to solve than verbal problems for both the arithmetic and algebra problems.

Verbally presented start-unknown problems (e.g., Problems 1 and 2) were considered particularly difficult. More than 76% of the teachers ranked story and word-equation start-unknown problems as more difficult than all other problem types. A post hoc comparison ($p = .05$) among all six problem types revealed that most teachers (70%) ranked start-unknown (algebra) word-equation problems as the most difficult form of problem given in the survey.

Researcher data. About 31% of the researchers consistently ranked symbolic-equation problems as easiest for arithmetic- and algebra-level problems. Only 23% of the researchers consistently ranked symbolic equations as harder to solve than word-equations and story problems within each of the two levels of the unknown-position factor. Like the teachers, the researchers tended to judge symbolic equations as easier to solve than verbal problems. However, researchers as a group were more likely than the teachers to judge symbolic-equation problems as difficult.

Relation to student performance. Contrary to teachers' expectations, students experienced greater difficulties when solving symbolic-equation problems than when solving verbally presented problems (Table 4). Also, in contrast to the views proffered by teachers and by researchers, students did not find algebra story and word problems to be most difficult. Instead, among algebra problems, symbolic-equation problems were significantly less likely to be correctly solved than either story problems or word-equation problems.

Interactions

More than 46% of the mathematics education researchers in our sample showed an interaction of the two factors by ranking (a) start-unknown (algebra) symbolic-equation problems as *easier* than start-unknown verbal problems and (b) result-unknown symbolic-equation problems as *harder* than result-unknown verbal problems (see Table 4). Only 17% of researchers ranked arithmetic symbolic-equation problems as being of the same difficulty as or harder than the two algebra verbal-problem types (e.g., Problems 1 and 2 of Table 1), whereas 69% ranked arithmetic symbolic-equation problems as being easier than either verbal start-unknown form. The remaining 14% of the participants were split in their predictions.

These data indicate that many mathematics education researchers view solving story problems as harder than solving word-equations or symbolic-equations for start-unknown problems and see all forms of start-unknown problems as harder than any result-unknown problems.

Summary

Teachers and researchers generally ranked algebra problems as more difficult than matched arithmetic problems, regardless of the presentation format. Teachers and researchers also tended to rank verbally presented problems (i.e., story and word-equation problems) as more difficult for students to solve than symbolic equations in algebra and arithmetic. Both professional communities in our samples generally agreed upon the relative difficulties of algebra story and word-equation problems.

DISCUSSION

Overall, teachers predicted much of what makes problems difficult for students. A Kendall's rank correlation shows a significant relationship between teachers' ratings and students' performances, $\tau(12) = .61$, $p = .03$ (see Table 4). However, the data also show systematic discrepancies with students' performances—discrepancies that could have significant implications for teachers' instructional practices. The most salient discrepancy is teachers' predictions that story problems and word-equation problems would be more difficult than symbol-equation problems, whereas students found symbolically presented problems most difficult. We now explore possible sources of this discrepancy by examining the structure of mathematics textbooks and their role in teachers' decision making. We then consider two competing models of algebraic development based on the student and teacher data sets.

Symbol Precedence View

In the course of the investigation of possible sources of teachers' and researchers' predictions, two commonly used mathematics textbook series adopted by the teachers' school district were analyzed for their treatment of arithmetic and algebra concepts (Nathan, 1998). Textbooks have been identified as a primary resource—and often the *only* source—of the content planning performed by expert and novice high school mathematics teachers (Borko & Livingston, 1989; Borko & Shavelson, 1990; Cooney, 1985). Both textbook series (Harcourt, Brace, & Jovanovich and Glencoe) included a prealgebra text and an Algebra I text. Both series first presented arithmetic computations in symbolic form, followed by the application of these procedures to stories and scenarios. The algebraic formalism was introduced next, along with rules of symbol manipulation and worked-out examples showing how the rules could be applied to problems. Story problems were then introduced as applications of the formalism. In the textbooks' solution to a story problem, the first step was translation of the verbal problem to a symbolic format, followed by application of symbol-manipulation procedures. The chapter organizations of the textbooks followed the general sequence shown in Table 4.

Kendall's rank correlation revealed a highly significant agreement between teachers' rankings (Table 4) and the curricular sequence offered by the mathematics textbook series, $\tau(12) = .867$, $p = .015$. Teachers ranked start-unknown (algebra) problems as significantly more difficult to solve than result-unknown (arithmetic) problems. They also ranked symbolic-equation problems within each category as easier than the corresponding verbal problems.

One likely interpretation of this pattern is that, through reliance on and repeated exposure to textbooks, teachers internalize the symbol precedence view as a basis for their predictions of problem difficulty for students. Alternatively, both teachers and textbook authors may base their views of the development of algebraic reasoning on a common perspective, such as the component view of mathematics learning (cf. Greeno, Collins, & Resnick, 1996).

Two Competing Developmental Models

The ranking data strongly indicate that teachers and researchers tend to view the development of students' algebraic reasoning in the following light. First, students develop their symbolic skills in arithmetic, with instruction and practice focused on result-unknown problems. Then, students learn to apply and extend these skills to verbally presented arithmetic problems. Next, students learn to take on more general families of problems in which the unknown quantity is not necessarily the result but can occur initially in the situation. Symbolic forms of these problems are presented initially because they are considered to be the simplest versions. For solving these problems, new procedures, concepts, and laws are introduced to support conventional symbol manipulation to isolate the unknown quantity as a means toward problem solving. Finally, these procedures are applied to the verbal format; students are taught translation and modeling rules so that the algebra word-equation and story problems are reduced to algebraic equations solvable by way of the symbol manipulation procedures acquired during the previous phase of instruction. Teachers, researchers, and textbook authors appear to think of students' problem-solving development largely within the symbol precedence view. Thus, teachers and researchers who are examining a problem for its level of difficulty make their decisions on the basis of the question "How far along the developmental trajectory from symbolic arithmetic to algebra story-problem solving has a student progressed?"

However, the analyses of students' problem-solving-process data suggest an alternative trajectory for the development of algebraic-reasoning ability, a trajectory that circumvents many of the difficulties of symbolic algebra. In this alternative view, verbal competence and the associated reliance on guess-and-test and unwinding strategies are hypothesized to precede symbol-manipulation skill for both arithmetic (result-unknown) and algebraic (start-unknown) problems.

With these two views in mind, one can frame students' performances and teachers' expectations in the form of two competing models of the development of algebraic reasoning. In one model, suggested by the analyses of textbooks and the teacher and researcher data, we hypothesize that students' symbolic-reasoning skills develop first, with word-problem-solving ability developing later. This hypothesis we refer to as the *Symbol Precedence Model* (SPM) of the development of algebraic reasoning. In the other model, based on the student data, we hypothesize that verbal reasoning precedes symbolic reasoning. This model we term the *Verbal Precedence Model* (VPM) of students' development of algebraic reasoning.

The SPM and VPM of students' development of algebraic reasoning are compared in Figure 3. Each square in the figure depicts the level of problem-solving competency that the student can achieve unaided (cf. the zone-of-proximal-development construct of Vygotsky, 1978). All 16 possible levels, with descriptive labels, are shown in Figure 3. The number in each level indicates the total number of students from the initial study (n = 76) who have progressed to that level but not beyond, as determined by their problem-solving performances. These numbers are used to compare, on the basis of the student data, the predictive power of the two models.

Figure 3 illustrates two pathways through this space of possible competency levels. One path is consistent with the SPM (dashed lines) of development, and the other (heavy lines) corresponds to the VPM of development. In the model that follows from the symbol precedence view (right of center), development of arithmetic before algebra and development of symbolic-problem-solving abilities over verbal reasoning are favored.

Note that three states of competency are held in common by these two competing models. Both models account for all problems being solved at the end of development and no problems solved ("None") at the beginning of development. In both models we posit that "All Arithmetic" competency occurs midway in the students' development.

According to both models, a student passes through the level of competency at which the student can solve all result-unknown problems regardless of their presentation formats. However, the two models differ in their predictions of the prerequisite competencies. In the VPM, verbal-arithmetic competency is hypothesized to precede result-unknown competency, whereas in the SPM, competency in symbolic arithmetic is hypothesized to occur first.

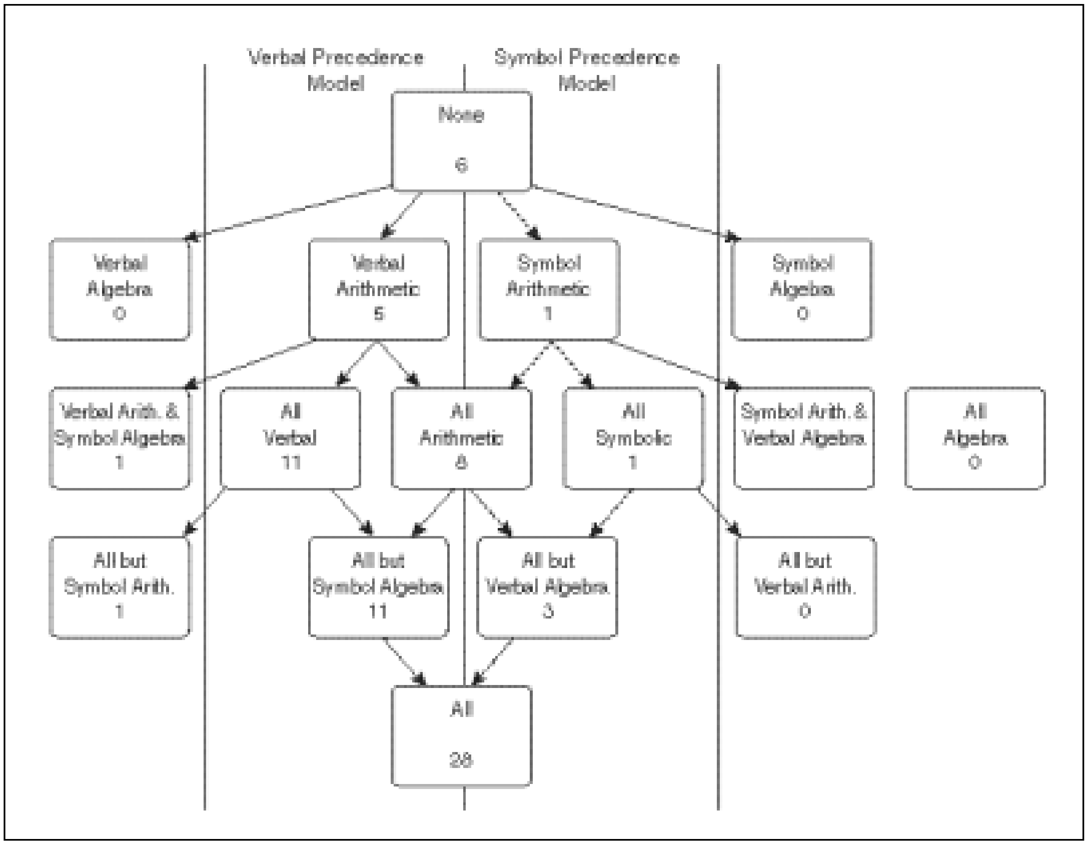

Figure 3. Two models of problem-solving ability (VPM with heavy lines and SPM with dotted lines) and their respective quantitative fits with the original student data (n = 76).

Quantitative Model Comparison

Using the patterns of student performance in the original study (n = 76), we classified each student into a competency level that shows the student's development. The domain can be characterized by 16 states of competency. Competing developmental models all work within these 16 states, but, in the two models, we hypothesize a different ordering that leads to skilled performance (the "All" state). States of competency may also be omitted in a model of development.

Students classified as competent in either none or all of the problem categories fit both models trivially (i.e., they received either a zero score or a perfect score on the assessment). A student's performance does not fit a model when the student occupies a state of the model that demonstrates one competency but lacks another that is hypothesized to precede it in the developmental sequence. On the one hand, if a student's competency reaches symbolic arithmetic but no further and the student cannot also solve verbal arithmetic problems, then that student fits the SPM but not the VPM because in the SPM competency in solving symbolic arithmetic problems is hypothesized to precede competency in solving verbal arithmetic problems. On the other hand, if a student's competency level reaches verbal-arithmetic problem solving but no further and includes no symbolic-arithmetic problem solving, she fits the VPM but not the SPM.

Alternatively, a student's competence may follow a trajectory different from that hypothesized by either developmental model. For example, a lone subject was found to have competence at solving verbal arithmetic problems and symbolic algebra problems but no others. According to both models, symbolic-arithmetic competence will precede symbolic-algebra competence. Consequently, performance of this subject conforms to neither model, reducing the predictive power of each.

As a quantitative measure of the predictive power of each of the two models, we found the percentage of students who follow each of the hypothesized trajectories (Figure 3). Of the 76 participants in the original study, 69 (about 91%) of the students fit the VPM, whereas 47 (62%) fit the SPM. Most of those who fit the SPM are the 42 (55%) students in the three levels common to both models (the central column). Only 5 students (7%) uniquely fit the SPM, whereas 27 (36%) uniquely fit the VPM. Two students (3%) remain outside both. According to the original student data, the VPM provides a better quantitative fit than the SPM.

As a further test the VPM and SPM were also applied to the data from the 171 students in the replication study. From that study, 151 students (88%) fit the VPM. In contrast, only 79 students (46%) fit the SPM. As before, most of those students who fall along the developmental trajectory of the SPM are the 65 students (38%) who are in the three competency levels common to both models. Only 14 students (8%) uniquely fit the SPM whereas 86 students (50%) uniquely fit the VPM, and 6 students (4%) remain outside of either model. This result provides added support to the hypothesis that the VPM of algebra development better reflects the problem-solving performance of students in the two samples than does the SPM. Furthermore, the VPM accounts for a high percentage of the students on an absolute scale, suggesting it captures something basic to the development of students' algebraic reasoning.

CONCLUSIONS

Although teachers accurately predict the differential performance of students on result-unknown and start-unknown problems, we found that students' problem-solving behaviors differ in systematic ways from those predicted by teachers and researchers. These differences have a significant effect on how teachers perceive students' reasoning and learning. In this final section, we explore some of these issues and consider their implications for research on students' thinking and for classroom instruction.

Implications for Research on Student Cognition and Development

Students' difficulties with symbolic problems, apparent in our data, challenge the oft-cited view (e.g., Cummins, Kintsch, Reusser, & Weimer, 1988; Mayer, 1982) that story problems are inherently harder than symbolic ones. A number of studies, mostly at the elementary and middle-grades levels, have shown positive effects of situational context (e.g., Baranes et al., 1989; Carraher et al., 1987; Cognition and Technology Group at Vanderbilt, 1993). Although the student data reported here show a similar pattern, we must note that there was a general advantage of *verbal* problems overall. Although a context advantage may exist as well (as the replication study suggests), in studies comparing story- and symbolic-problem solving, the intermediate case of word equations as a verbal-problem format without a context has been ignored, so comparisons reported here could not be made in those studies. Students in secondary education have developed their verbal-reasoning skills for a longer period of time than their skills with manipulating and reading mathematical-symbol structures. The accessibility and use of students' alternative solution strategies suggest a view of mathematical development that matches the development of students' verbal-reasoning abilities.

The student data also show that students can solve simple algebra problems as well as they can solve arithmetic problems when the arithmetic problems are presented symbolically and the algebra problems are presented verbally (with or without a context). Such algebra problems elicit from students powerful alternative strategies such as the guess-and-test and unwinding methods. These alternative strategies are general and fairly robust, providing, in many instances, ways for the solver to reduce error opportunities and to verify solution accuracy. The use of these alternative methods by college students who had not recently studied algebra has also been observed (Kieran, 1992; Tabachneck et al., 1994), suggesting that the methods are accessible and resistant to extinction—unlike algebraic symbol-manipulation skills, which were poorly executed by the algebra students in our samples.

Implications for Research on Teacher Cognition

Understanding teachers' views of problem difficulty is of value because these beliefs are likely to affect teachers' instructional planning and the design of their assessments (cf. Borko & Shavelson, 1990; Carpenter et al., 1980). If teachers misperceive the relative difficulties of symbolically presented problems, they may choose to withhold verbally presented problems from a struggling student, with the rationale that verbal problems are simply out of reach for the student. Our results suggest that the development of students' algebraic reasoning and problem solving must be examined more closely before teachers make curricular decisions. This suggestion is especially important when school districts throughout the United States are exploring ways to teach students algebra in the primary grades.

Mathematics educators need to be made aware of the efficacy and flexibility of students' alternative reasoning strategies. We suggest, however, that our findings reflect deep-seated views held by mathematics educational researchers and some textbook authors as well as high school teachers. Any attempts to address this issue will need to explicitly address teachers' belief systems that support these views. As mathematics education researchers, we must remember that deep-seated beliefs do not easily change. If we want to ultimately bring teachers' views into closer alignment with empirical findings, we must make teachers aware that they hold these views. These views must be explicitly characterized for teachers; the strengths must be acknowledged and the limitations exposed.

Implications for Mathematics Instruction

The finding that algebra learning is formidable for students is not news. However, insights into how students cope with their attempts to solve new problems, learn formal representational systems, and think in new ways are of great interest. Students, in employing informal strategies, demonstrate an intuitive understanding of quantitative relations that may prove advantageous in later instruction.

Several researchers have reported the advantages of explicitly addressing these informal methods during instruction and building on them as a means to teach formal algebraic methods. Research has shown problem-solving performance advantages for students who use informal and formal methods in combination (Koedinger & Tabachneck, 1994; Petitto, 1979). The advantage of using multiple strategies often is viewed in a compensatory manner; that is, the weaknesses of one strategy are offset by the strengths of another (Koedinger & Tabachneck, 1994). Guess-and-test and other substitution methods have proven to be beneficial for students who are developing their understanding of a balanced equation (Kieran, 1988). These methods apparently helped the students see equations structurally, providing a view that ultimately facilitated students' acquisition of formal methods that involved performing equivalent operations on both sides of the equation.

For algebra instruction, a recently implemented alternative approach builds on sixth graders' facility with these informal solution methods to teach the concepts and procedures of the formal approach of algebra problem solving (e.g., symbolizing of situations, manipulating of symbolic expressions, solving systems of linear equations, and identifying and representing pattern generalizations) (French, 1999; Knuth, 1999; Koedinger & Alibali, 1999; Nathan, 1999). In this approach, the unwinding and guess-and-test methods used spontaneously by students served as grounding representations for the new, more abstract objects and procedures. Key to this instructional approach was the use of diagnostic pretests to identify students' methods and inform bridging instruction aimed at connecting the new concepts to the learner's prior knowledge (cf. Kaput, 1989). Two studies, one using animations of situations (Nathan et al., 1992) and one using concrete arithmetic instances (Koedinger & Anderson, 1997), have shown that instruction that bridges formal algebra instruction to previously grounded representations helps students learn processes such as algebraic modeling of verbally presented relations. The two studies differed in the type of grounded representations used, yet they yielded similar results, suggesting that a crucial feature of success was the role of grounded intermediate representations in students' learning.

Approaches such as those briefly reviewed address curricular goals that encourage middle-grades students to "develop and apply a variety of strategies to solve problems" (National Council of Teachers of Mathematics, 1989, p. 75) and solve equations "using concrete, informal, and formal methods" (p. 102) so that they will "develop technical facility" (p. 150) with the later concepts of algebra. However, the interpretation of the use of informal methods must be cautionary as well as optimistic. Informal methods may facilitate performance when technical facility is lacking, but informal strategies can be limited in their efficacy. For example, the unwinding strategy breaks down when an unknown quantity has multiple occurrences. Likewise, the guess-and-test strategy can be inefficient, highly demanding of cognitive resources, and limited to finding numerical answers that are likely to be guessed (e.g., whole numbers and common fractions) (Tabachneck, Koedinger, & Nathan, 1995).

Informal strategies also show their limitations as problem complexity increases. Verzoni and Koedinger (1997) found that middle school students (Grades 6 through 8) performed best on easy (one-operator) problems when they were presented in a grounded story-problem format, rather than as an abstract number sentence, because the story problems elicited more successful informal strategies. However, when students were given more complex problems that involved two operators and the use of negative numbers, performance was higher on formal number-sentence problems because the greater complexity interfered with the execution of the informal solution methods.

Informal methods have also been shown to inhibit the acquisition of formal algebraic solution strategies in an instructional setting. Although guess-and-test users in Kieran's (1988) study were more apt to learn how to isolate unknown terms by maintaining a balanced equation, the students who preferred to use working-backwards methods like unwinding had greater learning difficulties. Learning suffered because working-backwards methods seemed to reinforce a procedural view of solving algebraic equations instead of supporting a structural view.

In conclusion, as research into mathematical learning and instruction continues, teachers and members of the research community will be provided with greater understanding of students' mathematical conceptions and development. And as studies of teachers' knowledge and beliefs continue, enhanced programs of teacher preparation and the development of theoretically and empirically rooted approaches to classroom instruction are to be expected.

REFERENCES

Anderson, J. R., Reder, L. M., & Lebiere, C. (1996). Working memory: Activation limitations on retrieval. *Cognitive Psychology, 30,* 221–256.

Baranes, R., Perry, M., & Stigler, J. W. (1989). Activation of real-world knowledge in the solution of word problems. *Cognition and Instruction, 6,* 287–318.

Borko, H., & Livingston, C. (1989). Cognition and improvisation: Differences in mathematics instruction by expert and novice teachers. *American Educational Research Journal, 26,* 473–498.

Borko, H., & Shavelson, R. (1990). Teacher decision making. In B. F. Jones & L. Idol (Eds.), *Dimensions of thinking and cognitive instruction* (pp. 311–346). Elmhurst, IL: North Central Regional Educational Laboratory and Hillsdale, NJ: Erlbaum.

Briars, D. J., & Larkin, J. H. (1984). An integrated model of skill in solving elementary word problems. *Cognition and Instruction, 1,* 245–296.

Carpenter, T. P., Corbitt, M. K., Kepner, H. S., Lindquist, M. M., & Reys, R. E. (1980). Solving verbal problems: Results and implications from national assessment. *Arithmetic Teacher, 28*(l), 8–12.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1994). *Cognitively guided instruction: Children's thinking about whole numbers.* Madison: Wisconsin Center for Education Research.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1985). Mathematics in the streets and in schools. *British Journal of Developmental Psychology, 3,* 21–29.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1987). Written and oral mathematics. *Journal for Research in Mathematics Education, 18,* 83–97.

Carraher, T. N., & Schliemann, A. D. (1985). Computation routines prescribed by schools: Help or hindrance? *Journal for Research in Mathematics Education, 16,* 37–44.

Cognition and Technology Group at Vanderbilt. (1993). The Jasper experiment: Using video to furnish real-world problem-solving contexts. *Arithmetic Teacher, 40,* 474–478.

Cooney, T. J. (1985). A beginning teacher's view of problem solving. *Journal for Research in Mathematics Education, 16,* 324–336.

Cummins, D. D., Kintsch, W., Reusser, K., & Weimer, R. (1988). The role of understanding in solving word problems. *Cognitive Psychology, 20,* 405–438.

De Corte, E., Greer, B., & Verschaffel, L. (1996). Mathematics teaching and learning. In D. C. Berliner & R. C. Calfee (Eds.), *Handbook of educational psychology* (pp. 491–549). New York: Simon & Schuster Macmillan.

French, A. (1999, April). *A teacher's perspective on being the object of study in research on teacher cognition and instructional practice.* Paper presented at the annual meeting of the American Educational Research Association, Montreal, Canada.

Graves, B., & Zack, V. (1996). Discourse in an inquiry math elementary classroom and the collaborative construction of an elegant algebraic expression. In L. Puig & A. Guteiérrez (Eds.), *Proceedings of the 20th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 27–34). Valencia, Spain: University of Valencia.

Greeno, J. G., Collins, A. M., & Resnick, L. (1996). Cognition and learning. In D. C. Berliner & R. C. Calfee (Eds.), *Handbook of educational psychology* (pp. 15–46). New York: Simon & Schuster Macmillan.

Guberman, S. R. (1987, April). *Arithmetic problem solving in commercial transactions of Brazilian children.* Paper presented at the meeting of the Society for Research in Child Development, Baltimore.

Hall, R., Kibler, D., Wenger, E., & Truxaw, C. (1989). Exploring the episodic structure of algebra story problem solving. *Cognition and Instruction, 6,* 223–283.

Kaput, J. J. (1989). Linking representations in the symbol systems of algebra. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 167–194). Reston, VA: National Council of Teachers of Mathematics and Hillsdale, NJ: Erlbaum.

Kieran, C. (1988). Two different approaches among algebra learners. In A. F. Coxford (Ed.), *The ideas of algebra, K–12.* 1988 Yearbook of the National Council of Teachers of Mathematics (pp. 91–96). Reston, VA: National Council of Teachers of Mathematics.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 390–419). New York: Macmillan.

Kieran, C., & Chalouh, L. (1990). *Prealgebra: The transition from arithmetic to algebra.* Unpublished manuscript, Université du Québec à Montréal, Canada.

Knuth, E. (1999, April). *The study of whole classroom mathematical discourse.* Paper presented at the annual meeting of the American Educational Research Association, Montreal, Canada.

Koedinger, K. R., & Alibali, M. (1999, April). *A developmental model of students' early algebra competency.* Paper presented at the annual meeting of the American Educational Research Association, Montreal, Canada.

Koedinger, K. R., & Anderson, J. A. (1997). Illustrating principled design: The early evolution of a cognitive tutor for algebra symbolization. *Interactive Learning Environments, 5,* 161–180.

Koedinger, K. R., & Nathan, M. J. (1998). *The real story behind story problems: Effects of representations on quantitative reasoning.* Manuscript submitted for publication.

Koedinger, K. R., Nathan, M. J., & Tabachneck, H. J. M. (1996). *Early algebra problem solving: A difficulty factors analysis* (Tech. Rep.). Pittsburgh, PA: Carnegie Mellon University.

Koedinger, K. R., & Tabachneck, H. J. M. (1994, April*). Two strategies are better than one: Multiple strategy use in word problem solving.* Paper presented at the annual meeting of the American Education Research Association, New Orleans, LA.

Koedinger, K. R., & Tabachneck, H. J. M. (1995, April*). Verbal reasoning as a critical component in early algebra.* Paper presented at the annual meeting of the American Educational Research Association, San Francisco.

MacLane, S., & Birkhoff, G. (1967). *Algebra.* New York: Macmillan.

Mayer, R. E. (1982). Different problem-solving strategies for algebra word and equation problems. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 8,* 448–462.

Nathan, M. J. (1998). *An analysis of the presentation of arithmetic and algebraic topics in common mathematics textbooks.* Manuscript in preparation.

Nathan, M. J. (1999, April). *An instructional theory for early algebra that incorporates research on student thinking, teacher beliefs, and classroom interactions.* Paper presented at the annual meeting of the American Educational Research Association, Montreal, Canada.

Nathan, M. J., Kintsch, W., & Young, E. (1992). A theory of algebra-word-problem comprehension and its implications for the design of learning environments. *Cognition and Instruction, 9,* 329–389.

Nathan, M. J., Knuth, E., & Elliott, R. (1998, April). *Analytic and social scaffolding in the mathematics classroom: One teacher's changing practices.* Presentation at the annual meeting of the American Educational Research Association, San Diego, CA.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

Petitto, A. (1979). The role of formal and non-formal thinking in doing algebra. *Journal of Children's Mathematical Behavior, 2*(2), 69–82.

Polya, G. (1957). *How to solve it: A new aspect of mathematical method* (2nd ed.). Princeton, NJ: Princeton University Press.

Rachlin, S. L. (1989). The research agenda in algebra: A curriculum development perspective. In S. Wagner & C. Kieran (Eds*.), Research issues in the learning and teaching of algebra* (pp. 257–265). Reston, VA: National Council of Teachers of Mathematics and Hillsdale, NJ: Erlbaum.

Riley, M. S., Greeno, J. G., & Heller, J. I. (1983). Development of children's problem-solving ability in arithmetic. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 153–196). New York: Academic Press.

Saxe, G. B. (1988). Candy selling and math learning. *Educational Researcher, 17*(6), 14–21.

Schoenfeld, A. H. (1998). Toward a theory of teaching-in-context. *Issues in Education, 4,* 1–94.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher 15*(2), 4–14.

Tabachneck, H. J. M., Koedinger, K. R., & Nathan, M. J. (1994). Toward a theoretical account of strategy use and sense-making in mathematics problem solving. *Proceedings of the 1994 annual conference of the Cognitive Science Society* (pp. 836–841). Hillsdale, NJ: Erlbaum.

Tabachneck, H. J. M., Koedinger, K. R., & Nathan, M. J. (1995). An analysis of the task demands of algebra and the cognitive processes needed to meet them. In *Proceedings of the annual meeting of the Cognitive Science Society* (pp. 318–327). Hillsdale, NJ: Erlbaum.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 127–146). New York: Macmillan.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. F. Coxford (Ed.), *The ideas of algebra, K–12.* 1988 Yearbook of the National Council of Teachers of Mathematics (pp. 8–19). Reston, VA: National Council of Teachers of Mathematics.

Usiskin, Z. (1997). Doing algebra in grades K-4. *Teaching Children Mathematics, 3,* 346–356.

Verzoni, K., & Koedinger, K. R. (1997, March). *Student learning of negative number: A classroom study and difficulty factors assessment.* Paper presented at the annual meeting of the American Educational Research Association, Chicago.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes.* (M. Cole, V. John-Steiner, S. Scribner, & E. Souberman, Eds.). Cambridge, MA: Harvard University Press.

Authors

Mitchell J. Nathan, Assistant Professor, School of Education, University of Colorado, Campus Box 249, Boulder, CO 80304; mitch.nathan@colorado.edu

Kenneth R. Koedinger, Senior Research Scientist, Human-Computer Interaction Institute, Carnegie Mellon University, Pittsburgh, PA 15213; Koedinger@cmu.edu