CHAPTER 4

UNDER THE MICROSCOPE OF RESEARCH AND INTO THE CLASSROOM: REFLECTIONS ON EARLY ALGEBRA LEARNING AND INSTRUCTION

Amy French
Angevine Middle School

Mitchell J. Nathan
University of Wisconsin

In the spring of 1995, when Mitchell Nathan first approached me about being part of an early algebra study, I was intrigued. I had recently spent several months with Boulder Valley School District's algebra study group—educators dedicated to fostering the growing mathematics reform movement by providing elementary, middle and high school students with meaningful access to the study of algebra. Mitch's proposal seemed like a natural next step in my teaching goals. I pondered over my final decision, however, because I had concerns about the time commitment. I was consumed by my work—why would I want to take on anything extra? I already knew I was a good teacher, highly respected in the community for my dedication to students and effective teaching style. My evaluations were outstanding, and I had a file full of letters from grateful parents. A little, nagging fear began to emerge that maybe I would not pass muster under

Teachers Engaged in Research, pages 49–68

the university microscope. However, I ignored the disquietude and said yes. Why? Despite some minor mathematical insecurities, I felt confident enough about my practice to tackle a challenge. It had been five years since I had made any significant changes in my teaching, and I understood the value of change for the sake of growth. I saw myself teaching for only another ten years or so, and I wanted to make something of those last years. I was just beginning to see the value of the math reform movement, and needed some fresh strategies. Research sounded important and meaningful, and Mitch's initial approach to me was open, honest and collaborative.

Mitch had certain stated goals that guided the project. He and his colleagues sought to understand how students used their own intuitions during problem-solving situations in the classroom and to determine how this type of understanding could be used by teachers to facilitate the learning of formal algebraic concepts and problem-solving methods. By addressing both students' informal conceptions of algebra and teachers' beliefs about students, they hoped to develop practical ways to improve algebra instruction in middle school level classrooms. These goals shaped three specific research objectives:

1. To document students' informal algebra problem-solving strategies and discourse before students have received formal algebraic instruction;
2. To document teachers' beliefs about students' intuitive solution methods, understand how these beliefs affect teachers' instructional practices, and observe how teachers' beliefs and practices change as they learn more about the effectiveness of students' informal solution methods; and
3. To understand how knowledge of students' informal solution methods can leverage the use of more formal and more general methods for solving algebra level problems, and observe how teachers draw on new understandings of students to foster algebraic reasoning and discourse in the classroom.

Several forms of data were collected over the years. The research team examined students' written work on algebra-level problems carefully designed to allow comparisons between problems in different formats (such as matched equations and story problems). Student solutions were then analyzed for their strategies and representations and the nature of their errors. (These analyses were later published. Relevant papers for the interested reader are Nathan and Koedinger 2000a, 2000b; Koedinger and Nathan, 2004.) In addition, the team collected video tapes of classroom interactions and student discourse in small and large groups, as well as students' presentations and justifications of their solution methods. (Relevant papers include Nathan and Knuth, 2003; Nathan et al., 2002.) The research team also conducted frequent interviews with the teacher to help uncover her own rationale for classroom practices (A relevant paper is Nathan and Knuth, 2003). Finally, to obtain a broader perspective, the team administered surveys to elementary, middle and high school teachers designed to elicit their beliefs of students' mathematical abilities and development. (A relevant paper for this is Nathan and Koedinger, 2000c.)

UNDER THE MICROSCOPE

We began meeting in June of 1995, in Mitch's small office at the University of Colorado in Boulder. Crammed around a small table, I was mentally poked and probed by Mitch and two assistants, Rebekah Elliott and Eric Knuth, both doctoral students in mathematics education. From the very beginning, I never minded being questioned by Mitch. I always felt safe with him. Even though he had a mathematics degree, I felt that his main interest lay in the research. But explaining myself to the graduate students was nerve-wracking. I soon realized that being concerned about time had been only part of the reason for my hesitation to agree to this project. Those little nagging fears about my abilities grew. My minor mathematical insecurities became major. I was afraid to be found out that I lacked mathematical confidence and knowledge. I had struggled with mathematics all of my life, did not have a mathematics degree, and only by chance had become the lead mathematics teacher on a sixth-grade teaching team. While it was no secret to the school district, my principal and my colleagues that my qualifications did not include a mathematics degree (my certification is in elementary education), I was sure it would become a problem. How could I possible talk intelligently about mathematical concepts? My last math class had been a Marilyn Burns workshop in 1989 and prior to that basic college algebra in 1969!

That first summer, we talked (or rather I talked) about myself and my beliefs about teaching and learning. I rather enjoyed this at first. I explained why I had become a teacher, what made a good teacher, what was hard about teaching, and so on. I proudly described my teaching history, the horror stories of the first year, all the subjects I had taught (by this time I had experience with every discipline). My confidence and pride lessened somewhat when it began to feel like a therapy session. I was probed (using the method described by Fenstermacher and Richardson in Richardson, 1994) to explain why I felt and acted the way I did. What were my core beliefs? Why did I believe them? It seemed like every response of mine brought on another onslaught of "why." I would like to think that I was totally honest about my beliefs, hopes and fears during that first round of questioning, but I know that I felt compelled to withhold the big one—my perceived lack of mathematical knowledge. I did admit to feeling vulnerable and insecure, and revealed my own fears about mathematics as a child and young adult. I remember I even explained that one of the reasons I wanted to teach mathematics for understanding was because I did not have that experience in my own schooling, and I felt it would have made a difference in my attitudes and success with mathematics. Despite these confessions, it took time before I would admit to the level of my mathematically inadequate feelings.

In summary, I enjoyed the questioning that probed my memories and my attitudes about my early years and teaching in general. I liked the self-reflection and new insights it brought because I was confident about my instructional style and understanding of sound pedagogy. However, when the questions became more focused on the mathematics, I felt the same kind of nervousness and angst as I did when I was a child, trying to unravel a word problem and panicking that I would be unable to find the answer. That would mean I was “dumb.” In that first summer of research, I did not want to appear dumb.

When school started, there was new pressure—the video camera. Now, I did not mind being filmed at first. I am a performer and I love an audience, so I was not particularly self-conscious during the taping. But I hated watching myself. I was so afraid that I would see myself saying something mathematically incorrect or doing something pedagogically unsound. Maybe my students would be difficult to manage that day, and I would be perceived as having poor classroom management (every teacher’s nemesis). However, I learned something about myself that I had always suspected. I knew my students well, and I could often assess what had happened in class without even seeing the tape. As a result, viewing the videos became less of an embarrassment and more of a test to see if what I thought had happened in class was accurate. Once past the self-consciousness, I was able to acknowledge my successes and analyze critically lessons without cringing.

Mitch, Eric, Rebekah and I continued to meet once a week to analyze tapes and discuss my teaching. Questions continued to become more mathematically pointed. Why did I do that? What were my teaching goals? What were my mathematical goals? What did I want the students to know? I vacillated between feeling that I had already answered their questions (a million times already—why did they keep asking me?) and secretly knowing that I was somehow avoiding the real substance of their questions. Could I continue to fake my (perceived) lack of mathematical understanding?

I cannot remember now whether or not there was a specific moment in time when a breakthrough occurred in my ability to feel safe enough to answer more honestly and not feel ashamed by asking questions. I suspect that it happened over time, but I believe that Mitch was largely responsible. His questioning style, while thorough and keen (and sometimes relentless!), was also gentle and sensitive. He made it apparent that he truly cared about the honesty of my answers. No matter what I said, it was valued. He was infinitely patient. I can remember long silences in our conversations when he would ask me to "say a little more" about a feeling or belief. Self-disclosure became not only easier, but also necessary for me. I became fascinated by my inner feelings, and couldn't wait to unearth every memory, detail and revelation. It occurred to me that in the early stages of the project I had feared that I was being rated and judged by my answers, when in reality my open and honest disclosures were essential to the basis of the research. I distinctly remember feeling that by the second summer of inquisition, I was more confident and felt more like an equal participant.

There were several foci that summer of 1996 and while we still continued to fine tune the basis for my beliefs, discussion centered more on my classroom practices and mathematical goals. We began looking at videotapes and writings of other educators. I enjoyed not being the center of attention for a while, and feasted on the rich ideas and practices of Vicki Zack, Deborah Ball and Jim Minstrell. I felt that the relationship among the research team members change somewhat. I thought, "Okay, now you know what I believe, and you've seen me teach for an entire year. How can we make it better?" The desire to improve my teaching became our common mission, and it was very exciting. It was inconsequential that I didn't consider myself a mathematics expert because my mathematical prowess was not the focus. They needed me to help design a teaching plan for the research, and I needed them to achieve my goal of becoming a better teacher. We all had an equal stake in the outcome.

For the next two years, my role as an object of study continued to grow and evolve. I felt increasingly responsible for my contribution to the research. It wasn't just Mitch's research anymore; it was ours. My expertise as the classroom teacher became more and more important as we developed specific lesson plans and activities to engage students in meaningful learning. I had to make decisions about whether an activity was captivating, reasonable in content and length and appropriate to the skill level of my students, and in addition how it addressed my mathematical goals. It became apparent that I had mathematical skills, and that I had really been selling myself short. By the summer of 1998, I felt as though Mitch and I had solidified a true partnership. He had arrived in 1995 with a lot of plans, and he needed a classroom in order to try them out. I had the classroom, and was hungry for new ideas. This collaboration exceeded our expectations. Was it worth the long summers of extra work? Was it worth the discomfort of self-disclosure? Has it changed my teaching practice in a positive way? My answers are yes, yes and yes.

INTO THE CLASSROOM

Earlier, I stated that I had been a successful and well-respected teacher prior to agreeing to participate in the early algebra research project at the University of Colorado. What about my teaching did I want to change? In attending various math workshops and reading articles in journals, it was apparent that one of the main concerns of reform educators and the National Council of Teachers of Mathematics [NCTM] was that students did not seem to have deep understandings of mathematical concepts. Many were able to memorize procedures and do calculations, but were unable to use this knowledge in practical applications or in problem solving. I had seen plenty of this in my own practice, and was eager to make mathematics more understandable even prior to meeting Mitch. I began to look for ways I could tell my students why something was true—ways that I could demonstrate proof. I reasoned that if I could show why, for example, the identity property could generate equivalence, students would better understand the concept of simplifying fractions. Even with repeated demonstrations and examples, I was having limited long-term success with student understanding. I felt I needed help.

During the first two summers of research, I maintained that a core belief of mine was that human beings learn by doing. Not far behind that belief was another one that students often learn from each other—sometimes better than from their teachers. Armed with those tenets, I began calling on students to explain their thinking to others. My uncertainty and insecurity with how to manage this is portrayed in a quote from one of our early classroom tapes in 1995. Inviting a student (all students are referred to by pseudonyms) to the overhead to show her solution to a problem, I asked, “Maggie, would you be willing to just sort of explain yours?” When I reviewed the vague and unconvincing phrase “sort of explain” on the video, I was even more determined to become more proficient in incorporating student presentation and discourse into my practice.

This was when we shifted the focus from me to other educators. I viewed several of Vicki Zack’s (1996) classroom videotapes, and began incorporating many of her ideas into my own classroom. Using small groups to facilitate student interaction, I posed problems such as “Washing Hair” (Meyer, 1983), and “Identifying Qualitative Graphs” (van Dyke, 1994), listened to students’ ideas as they worked together, and then had students present their solutions to the whole class. I stopped using phrases like “sort of explain” and began to ask students to be clearer about their thinking. I also incorporated more writing as a way for students to explain their reasoning. One problem that worked particularly well for this was from the NCTM (1992) Addenda Series *Geometry in the Middle Grades.* Students are asked to view a series of pictures based on a map. The writing prompt was "Unfortunately the pictures were dropped and got mixed up. Can you put them in the right order? Explain your thinking." Another example was, "How can five pizzas be divided equally among three friends? Explain your thinking using words and pictures." I liked the new direction, and found that some of the most interesting mathematical ideas surfaced and misconceptions were flushed out as I learned how to ask better questions of my students.

I still felt as though I was doing too much talking. I wanted my students to begin asking the questions of each other that I was currently posing. I wanted them to really listen to each other, and to feel safe in speaking to the group. In keeping with my basic tenets of learning from one another and learning mathematics by doing mathematics, it was important to me that my students learned to speak directly with each other. Deborah Ball's (1990) tape on "Shea Numbers" was a true inspiration. After Deborah led some routine class discussion, one of her students posed a theory about odd and even numbers. What ensued in her class was what I had been dreaming of. A student named Shea shared a hypothesis about a mathematical concept, and a spirited discussion followed, led by the students' own arguments, examples and counter-examples. The students were respectful to each other and very engaged. Deborah facilitated this by staying in the background, encouraging different students to share their ideas, and asking good questions that forced students to clarify their opinions. How could I get my classroom to look like that?

The research team discussed the elements of a classroom committed to discourse. We realized that we would need to have conversations with students starting from their first day that this would be a class where they would be expected to talk and listen. We used desk arrangements that allowed all the students to see each other during whole group discussions, but easily converted to an arrangement for small group discussions. We consulted with Lew Romagnano, professor of mathematics education at Metro State College in Denver, about meaningful problems, and questioning and assessment techniques. We selected and developed problems that did not lend themselves to simple calculations, but depended on multistep problem solving and had multiple entry points and solution paths. We also worked on establishing an atmosphere of trust and safety for students to speak publicly to the class, the teacher, as well as members of one's group. To help with this, we asked Laura Till, a mediator and professional facilitator with experience in mathematics education, to be a consultant in the area of group dynamics. We spent two days with Laura doing non-mathematical, trust-building activities that allowed students to share things about their interests and home lives. Then we moved into problem-solving activities with a strong emphasis on hearing about everyone's approaches and students asking each other to explain or elaborate on their ideas. Students also reflected on their experiences collaborating with group members. The adults role-played common class dynamics that inhibited trust and sharing, such as interruptions, poor eye contact, non-constructive feedback, and weak presentations. Students then brainstormed how to correct these staged interactions. We added these contributions to a chart of good listening and speaking skills that was placed in the front of the room and referred to throughout the year. We also revisited these skills with Laura after the winter break.

Then our research group tackled a key issue: What were the big ideas about math that I wanted my students to understand? I had to start thinking about important, general concepts, not just topics or skills. For example, one truth I wanted my students to understand is that mathematics is not mysterious and magical; rather it has order and is logical. I tried to revisit that idea every time they generalized a pattern or made a mathematical connection. A few examples come to mind. The order of mathematics is beautifully illustrated by listing factor pairs of a number. I taught my students to realize that if they listed the pairs in order, they could tell when they had found them all, either by beginning to see repeated factors, or by recognizing that the product of the factors was becoming more "square." Later, during algebra instruction, my students learned that an equation was nothing mysterious because it followed logical steps in pattern generalization: draw a picture, create a table of values, describe the relationships, write an equation, and describe how the equation relates to the picture and the relationships.

Once the big ideas were in place, I could start matching them to relevant units of study and activities. This was a new approach for me. I had never really thought about general mathematical truths while crafting a lesson plan. Teachers are held so accountable for the list of topics that need to be covered, that they focus most of their attention on how to get through the curriculum. I was no exception.

Related to this is the tendency for many teachers to follow a curriculum guide blindly, regardless of what preconceptions and misconceptions their students might bring with them to the classroom about a given topic. Jim Minstrell's (1989) video about his pre-assessment of scientific concepts among his students and subsequent lessons and class discussions was very motivating. I began to take time to expose misconceptions and throw them out to the class for discussion. For example, a few common misconceptions among many sixth graders surface during discussions about division. Many believe that it is impossible to divide a smaller number by a larger one, or if you do, the answer will be negative. Challenging students' deeply held beliefs has led to many spirited discussions, like the one in Deborah Ball's class for which I was so desperate to have take place with my students!

All of these changes came about in the first two and a half years of the study. By the summer of 1998, Mitch and I were collaborating on specific early algebra instruction, which was the big focus of the project. Our meetings had evolved from inquisitions to shared decision-making sessions about how to best sequence and present algebraic concepts to sixth graders. I was in new territory, and I welcomed it. Instead of being embarrassed about something I saw on a video of myself, I was able to take an objective view and ask, "Now, what went wrong there? How could I have facilitated that more effectively?" Mitch and I would brainstorm ideas for revision. We went a step further. Instead of discussing what I was doing or not doing on the video, the focus shifted to what the students were learning or not learning. And instead of focusing on my insecurities, we discussed student learning in a mathematical context.

In the fall of 1998 we put our ideas to work. I was teaching at a middle school in the Rocky Mountain region that was predominantly Caucasian (86 %) with some Asian (6%), Hispanic (5%) and American Indian (2%) students. A small number of students (12%) qualified for free/reduced lunch and special education (13%). The mathematical performance of the students on the California Achievement Test (CAT) had an enormous range, from the 5th to the 99th percentile. During our collaboration, there were typically four to five students in each class who received special education support for their physical and cognitive disabilities. A paraprofessional served the classroom once a week to help meet these students' special needs.

Over the summer we developed a specific curriculum to address early algebraic thinking. The NCTM *Standards* call for algebra learning K–12, where students progress from being able to identify simple patterns to being immersed in formal algebra courses. Based on prior research in this project (e.g., Nathan & Koedinger, 2000b), we believed that middle school students had developed intuitive and informal methods to solve algebraic problems prior to formal algebra instruction, even though these topics were traditionally presented in later grades. However, rather than seeing them expressed in symbolic representations and formal procedures, these intuitive methods tended to be verbal in nature. Our approach focused on eliciting those verbal methods from students in the class, helping them to articulate and abstract them away from particulars of each problem, and then bridging them to more formal approaches. The following lesson on pattern generalization followed a two-week study of writing equations and their inverses to mathematically model written situations such as, "Barb was two years older than Sam." Students had also had some instruction on the order of operations. These topics fit perfectly with the NCTM *Standards* of algebra, patterns, and number and operations, and would extend students' earlier work with equations. I chose two problems that could be solved by discovering a numeric pattern and could also be modeled with objects or drawn on paper. My source *About Teaching Mathematics, a K–8 Resource* (Burns, 1992). I started with the "Toothpick Building" problem as presented ("How many toothpicks will you need to build a row of 100 triangles?") and then amended "A Row of Squares,"("If you line up 100 square tables in a row, how many people could be seated?") "A Row of Pentagons" and "The Banquet Table Problem" to create my version called "The Dinner Party Problem." The sequence of lessons lasted for approximately three class periods, or a total of 150 minutes.

THE TOOTHPICK BUILDING PROBLEM

On the first day of instruction, I handed the students a copy of "Toothpick Building." I spent some time explaining the pattern on the board before I turned students loose to work with their partners to accomplish two things—finish an incomplete table of values and describe, using words and pictures, the relationship between the number of toothpicks one needs to build a given number of triangles strung together in a row of varying length. My initial goal was for students to get their verbal descriptions of the relationship on the board so that they could see easily each other's thinking and we would have a basis for class discussion.

Our observations with students in previous years (Koedinger & Nathan, 2004; Nathan & Koedinger, 2000a, 2000b) led us to believe that verbal forms of reasoning and representations were natural entry points for students learning to develop skills in algebraic modeling. Prior research also has shown that students have more success using verbal representations to express quantitative relations and solutions than they do with more formal representations including equations and graphs (Koedinger & Nathan, 2004; Nathan & Koedinger, 2000b; Nathan et al., 2002). We drew directly on this evidence when we designed the instructional approaches for these pattern generalization problems, consciously using students' verbal reasoning abilities as a way to bridge to equations (Nathan, 1999).

I was hoping some students would go beyond the recursive method ("you add two toothpicks every time") that requires one to only think about the change of one of the variables (toothpicks) and, instead describe the pattern in terms of both the number of toothpicks and the number of triangles. I circulated among groups while Mitch and Kate Masarik (a doctoral student and research assistant who had joined the project) set up a microphone and video on one pair of students so we could gather data on their thinking. When all pairs of students had put their written explanations on the board, we were excited at the diversity of thought. While several students described the pattern recursively (as expected), there were three descriptions that showed students were attempting to key in on a relationship between both triangles and toothpicks.

A lively discussion ensued where students defended their explanations and looked for similarities and differences. I challenged them to describe how their description was like or unlike the others on the board. Almost every student had some observation to make and I attributed the high level of participation to the public display of all their work on the board. I then asked the students to write an equation that modeled the relationships described by their words. By this time, most had latched onto 2 x number-of-triangles + 1 = number-of-toothpicks as their equation of choice, but I wanted them to be able to test the equation using different values and explain *why* that equation was a reflection of the toothpick pattern. The common summary was "every triangle uses two toothpicks (2 x number-of-triangles) except for the first triangle which has three (+1)." In addition to this, one student was able to generate the inverse relation to find the number of triangles given a number of toothpicks. We were pleased with the students' responses to this first pattern problem and the general pedagogical process that encouraged so much student discourse and reflection. Thus, we were anxious to take their level of understanding step further.

THE DINNER PARTY PROBLEM

On day two, I had the students for a 90-minute block of time. Once again, Marilyn Burns was my source. I adapted "A Row of Squares" and gave it a context by telling students that I was planning a dinner party for an undetermined number of guests who would sit one person per side at square card tables connected to form a row of tables (see Figure 1a). Students continued to work in pairs and were given the charge of completing the table of values (see Figure 1b), writing a verbal description of the relationship between the number of tables and the number of people that could be seated (using words and pictures) and finally, writing an equation that matched their written description. Once again, we were rewarded with a variety of solution presentations from students.

Figure 1 - FPO

Figure 1. The Table Problem. (a) The arrangement of "guests" at the "dinner tables." (b) The student assignments along with an initial table of values for the pattern showing the number of guests that can sit down of you know the number of tables at the dinner party.

First, I asked each pair of students to read their verbal description orally and describe any diagrams that they used to understand the pattern. I charged the remainder of the class to ask the presenters for clarification when needed and to ask the presenters how the verbal description or diagram captured the pattern. Thanks to one pair of students, an opportunity arose to explore students' attempts at using relatively meaningless combinations of numbers in one column (number of tables) within the table of values to generate the value in the other column (number of people). As shown in Figure 2a, the verbal description "add the number of tables to the number below it then add one to get the number of people" works mathematically, but is completely divorced from the meaning of the variables in the Dinner Party problem. This is because it does not really model the causal or visual structure of the problem, it just uses the successive arrangements of the entries in successive rows satisfy the pattern. In Figure 2b, a student uses hand gestures to show that she combined numbers from adjacent rows to get the number of people. However, this account did not provide insight for algebraically modeling the problem situation.

Figure 2 - FPO

Figure 2. (a) A verbal description of the Dinner Party Problem. (b) A student gestures to show how combining values from successive rows in the first column (and then adding 1) gives the value in the second column.

In contrast, another student (see Figure 3) provided a verbal description and diagram that showed how with each successive table only two additional people can be seated. The first and last (end) tables, however, each provide seating for three guests. This discussion of students' verbal descriptions of the pattern lasted for about 20 minutes, with most students remaining engaged and on task, but I felt a stretch break was in order before we moved on to their equations.

Figure 3 - FPO

Figure 3. Another student explains how his verbal description models the dinner party.

When students returned from the break, we discussed their mathematical equations. The equations were more varied than were those given in the Toothpick problem. The pair of students from Figure 2 showed how their pattern of adding values in successive rows could be written symbolically (see Figure 4). They came up with $T + U + 1 = P$, where "U" represents "the number directly *under*" a value in the tables column.

Figure 4 - FPO

Figure 4. The equation that followed from the verbal description of Figure 2.

I suggested to these students that three variables (T, U and P) in one equation was confusing, and asked them if they could re-write "U" in terms of the number of tables (T). They could! It was agreed that U always equaled $T + 1$; that is, the number in the row below was the original number plus 1. The new equation became $T + (T + 1) + 1 = P$. Because of earlier equation work, students were able to combine like terms and generate T + T + 2 = P. It was then a logical step to produce 2 x T + 2 = P. Our discussion included several checkpoints where I asked students to test the equations by inserting values from our chart and confirm the equations were equivalent.

"That's great!" I told them, "but *why* does that equation fit the situation? How does it describe what is happening with tables and people?" Their conclusions were that there are always two people at the sides of every table (2 x T) and there is one person at each end (+ 2). This time, more than one pair of students wrote an inverse equation and tested it for accuracy (see Figure 5). For verification of understanding, I assigned a homework problem where students had to generalize a pattern and equation from hexagonal tables. Nine out of ten pairs of students were able to accurately describe the new pattern and write an appropriate equation.

Figure 5 - FPO

Figure 5. An example of student work showing the equation that modeled the Dinner Party Problem and its inverse equation.

In reviewing our tapes of this lesson, we concluded that there had been a multitude of process standards and principles at work. Our work emphasized the building of new mathematical knowledge through problem solving, particularly by having students apply and adapt strategies and reflect on their process through extensive communication. Students had to organize their thinking and present it to their peers verbally and in writing, where they were encouraged and coached to use the language of mathematics to communicate their ideas clearly, coherently and precisely. Their understanding of different representations was greatly enhanced, as witnessed by their more sophisticated use of verbal, spatial and symbolic notation with both pattern problems.

A large impetus for our involvement in lessons such as these was to address greater participation and involvement of students across the achievement spectrum. In looking back through our tapes, we discovered that *every* student in the class either wrote their thinking on the board or participated in the class discussion. We wanted to elicit students' intuitions and invented strategies that were clearly meaningful for them, and chose the partner/presentation model because of what we knew and believed about sixth graders as learners—they like to use visual representations and they like to talk to each other. This related directly to my core beliefs that were unveiled during our earlier summers of "teacher interrogation," where I stated that I felt children learn by doing and that students learn from each other. Our model of instruction also included frequent, diagnostic assessment. In fact, it was woven into the very fabric of the lesson planning, as we continually moved back and forth between pre-assessment, class discourse about student approaches, our verbally based theory of student mathematical reasoning and development, and the continual refinement of curriculum activities and instructional practices.

Despite the success of these lessons (as observed later on the video and by the students' positive performance on post-assessment), I experienced some of my old fears and insecurities while teaching, particularly when students would invent equations that I had not anticipated. It was occasionally unnerving to feel "on the spot," trying to analyze an equation quickly, believing I should understand instantly everything students were generating. Sometimes when I asked a student, "How does your equation work?" I was not sure of the answer myself, and once again would feel little prickles of inadequacy. But when I studied the classroom videotape later in the week, I felt that my questioning and teaching techniques used to help students flush out their ideas were very effective. For example, I frequently invited multiple responses to a question, asking with great frequency if there were other ideas or approaches. I consciously increased my wait time. I looked for occasions where I thought students would benefit from working things out on their own, or talking briefly to a neighbor. After just a few minutes into a new problem or activity I would often asked students to share with the whole class their initial ideas of how to start. I tried to foster the view that solutions did not speak for themselves, but needed explanations to be complete. I encouraged students to ask their peers questions before coming to me. I also asked students to wait until a person was done speaking before raising their hands, in order to give students the sense that they had the floor and did not have to rush their thinking. Finally, if I were uncertain about a student's idea or comment, I would always ask for clarification, or invite the student to show it on the board. Students had long, engaging discussions. They generated sophisticated equations. They were subsequently able to take their strategies from these very visual problems and apply them to more abstract patterns that did not have a context.

Prior to my participation in this research project, I am certain I would have approached this topic differently. First of all, I would not have spent the time to set up a classroom environment where students felt as free to explain their thinking. The preliminary work setting up classroom norms made students feel safe when asked to speak to me, their classmates, to Mitch, or into the microphone. Secondly, I probably would have guided students more in an effort to get to the heart of the generalization more quickly. We would not have taken the time to dissect the different equations, particularly if they did not work. I believe I would have given more direct instruction in the use of variables, instead of letting the concept evolve naturally. In general, I would have gone faster, done more talking, and not given my students as much credit for their innovative solutions, in an effort to cover the topic with greater efficiency.

EPILOGUE

Each of us, the researcher and the teacher, learned a great deal from this collaboration. From close and frequent observations of students, Mitch learned that students do indeed have powerful intuitions about mathematics that often do not form the basis of mathematics instruction. Even prior to instruction, students as young as sixth grade can reason well about algebraic relationships and can articulate these solutions when given the opportunity. Students may not be aware of the power and relevance of these methods, and may think that they did not do anything, or that they cheated by using methods such as guess and test.

From close collaborations and observations of a couple of teachers, Mitch learned that teachers face amazing challenges as they strive to satisfy the constraints imposed by curricular demands (including those imposed by standards), the needs of students, and the inflexible time demands of the school day and year. Mitch learned that teachers could be driven by overarching goals and tenets, such as valuing the ideas and opinions of everyone in class and even their own experiences as middle school students learning the mathematics for the first time. Mitch found that middle school mathematics teachers face unique challenges, since they are asked to take on more and more advanced areas of instruction (algebra is commonplace now in middle school though it was not so when we started our collaboration), though they often have not had a great deal of mathematics preparation (most he met were licensed as elementary educators, and had insecurities about middle level mathematics). But many middle school teachers were highly motivated to enhance their teaching practices and their understanding of mathematics, and welcomed professional development opportunities that provided greater content knowledge and occasions to reflect on their teaching practices.

For me, (Amy) the experience of being a classroom teacher involved in a research project has been rewarding, but not always easy. It was very difficult to admit that despite my sterling reputation, I needed to improve. It was unnerving to know that I was being taped doing a lesson I had never tried before, full of mathematical concepts that in the not-so-distant past had been daunting for me. Presently, it is even harder than ever to get through my curriculum, because I am so committed to my students understanding key ideas in great depth. Three years after the completion of this study, I changed schools. I am currently working at a very different middle school (still in the Rocky Mountain region). The school is almost three times larger, and has a greater percentage of minority, low-income, and second language students, with 30% Hispanic, 20% English-language learners, and 32% receiving free/reduced lunch. The current political climate, with its focus on teacher accountability and high-stakes testing, is predominantly oriented toward standardized test score gains. Too many of my current students have scored "unsatisfactory" on state mandated tests. Because of this, I am worried constantly about time and coverage. How can I possibly cover all they need to know in order to make positive growth on these assessments? I worry that my students will not make adequate gains and that future teachers will see gaps in their knowledge. It is time consuming to plan lessons that involve small group tasks and meaningful discourse. Fostering the discourse in class is a constant struggle. Kids are only used to talking to each other socially; they resist truly listening to a peer's academic thoughts. It is difficult for the second language learners to participate without a lot of scaffolding of the language beforehand. I really have to be prepared prior to class so that I can anticipate the possible directions the lesson will take. And during the lessons themselves, I cannot rest. I need to pose good questions, listen to students, challenge them to be clearer in their thinking, and keep the mathematics visible. It is exhausting.

However, as I look back on how I have evolved as a teacher and a learner, I know that as a result of this experience, I am a more confident and accomplished teacher. Instead of viewing the mathematical community with anxiety and feeling uncertain about my abilities, I am developing a critical eye and ear for authentic mathematical tasks. My goals for my students are much clearer, and what they master conceptually has become more important to me than the list of required topics. My students have opportunities to test out their ideas, engage in mathematical discussions, and present their findings in an environment that values their opinions and insights. But the most beneficial outcome has been the opportunity to engage in a high level of self-reflection. I agreed to participate in the original research project because I believed it would allow me to reflect on aspects of my instructional practices. In particular, I hoped to gain insights into students' mathematical intuitions about algebraic reasoning prior to formal instruction. I have learned more about teaching from this experience than any post-graduate class or in-service training. To have the time to truly think about one's core beliefs and subsequent practice, to have conversations with colleagues and professionals who value those beliefs, and then to successfully put these ideas into practice—this has been life changing.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the assistance, wisdom and inspiration of the following people: Rebekah Elliott, Eric Knuth, Kate Masarik, Lew Romagnano, Laura Till and Isobel Stevenson. The research discussed in this chapter was funded by the James S. McDonnell Foundation CSEP Program under the "Bridges for Representational Fluency" grant to Kenneth R. Koedinger, Mitchell J. Nathan, and Martha Wagner Alibali.

REFERENCES

Ball, D. (1990). *Shea numbers* [Motion Picture]. (Available from Mathematics and Teaching through Hypermedia, University of Michigan School of Education.).

Burns, M. (1992). *About teaching mathematics: A K-8 resource.* Sausalito, CA: Math Solutions Publications.

Koedinger, K R., & Nathan, M. J. (2004). The real story behind story problems: Effects of representations on quantitative reasoning. *Journal of the Learning Sciences, 13*(2), 129-164.

Meyer, C., & Sallee, T. (1983). *Make it simpler.* Palo Alto, CA: Dale Seymour Publications.

Minstrell, J. (1989). *James Minstrell's teaching tape* [Motion Picture]. Seattle, WA: Talaria.

Nathan, M. J., & Knuth, E. (2003). A study of whole classroom mathematical discourse and teacher change. *Cognition and Instruction, 21*(2), 175-207.

Nathan, M. J., & Koedinger, K. R. (2000a). Moving beyond teachers' intuitive beliefs about algebra learning. *Mathematics Teacher, 93*, 218-223.

Nathan, M. J., & Koedinger, K. R. (2000b). Teachers' and researchers' beliefs about the development of algebraic reasoning. *Journal for Research in Mathematics Education, 31,* 168-190.

Nathan, M. J., & Koedinger, K. R. (2000c). An investigation of teachers' beliefs of students' algebra development. *Cognition and Instruction, 18*(2), 209-237.

Nathan, M. J., Stephens, A. C., Masarik, D. K., Alibali, M. W., & Koedinger, K. R. (2002). Representational fluency in middle school: A classroom based study. In D. Mewborn, P. Sztajn, D. White, H. Wiegel, R. Bryant, & K. Nooney (Eds.), *Proceedings of the twenty-fourth annual meeting of the North American chapter of the International Group for the*

Psychology of Mathematics Education. Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Nathan, M. J. (1999 April). *An instructional theory for early algebra that incorporates research on student thinking, teacher beliefs, and classroom interactions.* Paper presented at the Annual Meeting of the American Educational Research Association, Montreal.

Richardson, V. (1994). *Teacher change and the staff development process: A case in reading instruction.* New York: Teachers' College Press.

van Dyke, F. (1994). Activities: Relating to graphs in introductory algebra. *Mathematics Teacher, 87*(6), 427-432, 438-439.

Zack, V. (1996). *Group of four: Washing hair* [VHS] (Videos taken by Author).