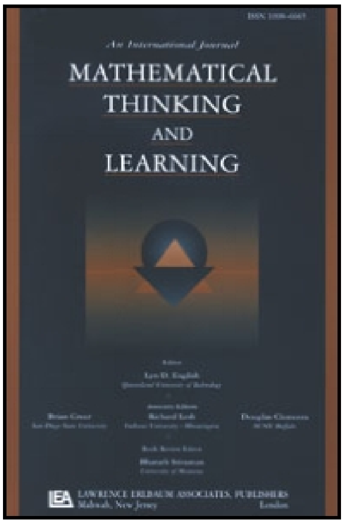

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Pattern Generalization with Graphs and Words: A Cross-Sectional and Longitudinal Analysis of Middle School Students' Representational Fluency

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Pattern Generalization with Graphs and Words: A Cross-Sectional and Longitudinal Analysis of Middle School Students' Representational Fluency

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Cross-sectional and longitudinal data from students as they advance through the middle school years (grades 6–8) reveal insights into the development of students' pattern generalization abilities. As expected, students show a preference for lower-level tasks such as *reading the data,* over more distant predictions and generation of abstractions. Performance data also indicate a *verbal advantage* that shows greater success when working with words than graphs, a replication of earlier findings comparing words to symbolic equations. Surprisingly, students show a marked advantage with patterns presented in a continuous format (line graphs and verbal rules) as compared to those presented as collections of discrete instances (point-wise graphs and lists of exemplars). Student pattern–generalization performance also was higher when words and graphs were combined. Analyses of student performance patterns and strategy use contribute to an emerging developmental model of representational fluency. The model contributes to research on the development of representational fluency and can inform instructional practices and curriculum design in the area of algebraic development. Results also underscore the impact that perceptual aspects of representations have on students' reasoning, as suggested by an Embodied Cognition view.

Representational fluency is growing in importance as the mathematics education community strives to reform instruction and curricula to provide students with learning experiences that expand beyond the narrow emphasis on equations (e.g.,

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Zazkis & Liljedahl, 2002). This shift is especially important as more students are introduced to algebraic reasoning as the study of mathematical objects and relations (e.g., Saul, 2001). There are many ways of representing numerical information, and students must learn to gain facility with a wide range of them. Two such representational forms are graphs and words.

Graphical representations are important to representational fluency because they show relationships spatially and extend students' range of quantitative representations and forms of reasoning. Graphs take on particular importance because they appear in a variety of fields outside mathematics, particularly in the physical and social sciences, where they are used to represent data and express theoretical relationships. Words are ubiquitous and are used to represent ideas and relations inside and outside the domains of math and science. However, words have always been important to mathematics and mathematics education, in part, because verbal representations can carry ideas across disciplines. Words are also very expressive, and facility with words helps people communicate their mathematical ideas and understand the ideas of others.

Representations can be in words or graphs, but they can also present patterns and functions as discrete (or digital) forms, or be continuous (or analog; see Case, 1985; 1992; 1996). One way to capture the discrete quality of a pattern is to show it as a collection of instances, such as points on a graph or a verbal list of examples, as when one collects data or conducts discrete trials of a simulation. The presentation of a pattern in a continuous form can include a graphical line or curve; or in similar fashion, as a verbal rule that portrays the entire relationship in a holistic manner. Thus, patterns can be depicted as a collection of related instances or holistically, in both verbal and graphical forms.

In this study we examined how different ways of presenting patterns affected the pattern generalization performance of middle school students. This was part of a larger aim of understanding the development of students' algebraic thinking (Tier 1) within the Supporting the Transition from Arithmetic to Algebraic Reasoning (STAAR) project (Nathan & Koellner, this issue). Our colleagues (Alibali, Knuth, Hattikudur, McNeil, & Stephens, this issue) explored other facets of representational fluency, particularly students' development of equal sign understanding and its relation to equation solving. In this article, our emphasis is on the uses of graphical and verbal representations. We were motivated by several general questions about the influences on US middle school students' (grades 6, 7, and 8) pattern-generalization performances and strategies. In addition to documenting developmental changes, we wanted to know:

1. How do different task demands affect students' performances?
2, How do students' performances differ when tasks are presented as discrete versus continuous patterns?
3. How do differences in representation influence students' performances?

REVIEW OF PRIOR RESEARCH

To explore these general questions, we will first selectively review the literature on representational differences in mathematical reasoning, particularly works about problem solving and generalization using words and graphs. We will include the influence of task differences that vary the demands placed on the student. We will also look at reported findings on the differences in students' reasoning and performance for patterns presented in discrete and continuous forms. This review will help us construct a preliminary model of the development of representational fluency, and revise these general questions so they reflect the current understanding of the field more accurately. The model will allow us to make specific predictions that can be tested with cross-sectional and longitudinal data analyses. The new findings that emerged from these analyses led us to propose a revised model of the development of representational fluency that includes influences of pattern presentation and task demands. Finally, we will discuss the implications of this model and our findings for teaching and learning and for future research on representational fluency.

Fluency with Graphs

A review of the literature reveals that children have many areas of confusion about the meaning and uses of graphs (Friel, Curcio, & Bright, 2001; Leinhart, Zaslavsky, & Stein, 1990). Some of the errors commonly made suggest children often have a poor understanding of the basic meaning of graphs as depicting relations between specific quantities. For example, children tend to interpret graphs literally, and may expect the shape of a graph to match the shape of the situation being represented (Clement, 1985; Monk, 1992; Smith, diSessa, & Roschelle, 1993) rather than a quantitative relation among values (such as speed and time). Other problems suggest a lack of understanding of the graphs' components (Beichner, 1994; Leinhardt, Zaslavsky, & Stein, 1990). Students can also project certain notions of linearity on graphical representations and impose a slope of one, axis scales of one (Lehrer & Schauble, 2001), or a zero intercept (Hadjidemetriou & Williams, 2000; 2001; Kaput & West, 1994). Students can also become fixated by the boundary frame of the graph, to the exclusion of the pattern represented within its bounds (Bieda & Nathan, 2006; Stevens & Hall, 1998). Taken together, these findings suggest that graphs are difficult for learners, and we can expect to see reduced performance on patterns presented graphically.

Fluency with Words

Past research on students' mathematical reasoning and development has also revealed the important role of verbal representations (Kaput, 1992; Nathan &

Koedinger, 2000). Koedinger and Nathan (2004) compared the performance and strategies of inner-city high school students ($N = 76$) who had successfully completed an algebra I course. They were given arithmetic and algebraic equations or matched problems using words. Verbal problems were either presented as story problems that included a situational context, or word equations that verbally described the relations found in symbolic equations, without an explicit problem context. Regardless of the representations used, all problems shared the same underlying quantitative structure. This design allowed Koedinger and Nathan (2004) to analyze performance differences between symbolic and verbal formats independent of context (word equation vs. equation), as well as the impact of context (story vs. word equation). They found that the high school algebra students demonstrated higher levels of performance (about 64% correct), solving the verbally presented story and word equation problems through the strategic application of highly reliable, invented solution strategies (such as unwinding and guess-and-test), while at the same time struggling to solve matched equations (getting about 43% correct; experiment 1). This *verbal advantage* has proven to be quite reliable across a range of populations and tasks, including other high school students ($N = 171$; Koedinger & Nathan, 2004, experiment 2), middle school students just learning algebra ($N = 90$; Nathan, Stephens, Masarik, Alibali, & Koedinger, 2002), community college students ($N = 153$; Koedinger, Alibali, & Nathan, in press, experiment 1), high-performing university students ($N = 65$; Koedinger et al., in press, experiment 2) and preservice teachers (Zazkis & Liljedahl, 2002; see also Knuth, Alibali, Weinberg, McNeil, & Stephens, 2005). Based on these findings, we expect to see a verbal advantage for pattern generalization when we compare performances with patterns presented using graphs and words.

Discrete and Continuous Representational Forms

Along with the representational differences reviewed above, we can compare how people reason about patterns that are presented as discrete or continuous. This is not an arbitrary distinction. Rather, this dichotomy parallels common psychological dimensions. Case's (1985; 1992; 1996) theory of how children develop understanding in any domain of study is that two initially separate but relevant types of understandings (primary mental schemas) are first developed in isolation and then become integrated through appropriate instructional experiences. One type of understanding is primarily sequential and digital, favoring words, numbers, and individual data points. In this form, a pattern may be a collection of instances. The other is spatial and analogical, and includes line graphs and illustrations. Here, the pattern, and its underlying relation, is presented in a continuous manner. Case's theory states that, in time, children can integrate these schemas; and when they do, their understanding of a domain is transformed and a new psychological construct is produced that underpins all current and further learning in the domain. However, early on, their reasoning is dominated by the separate forms of understandings mediated by the two different classes of formats.

Kupermintz and Nathan (2004) explored the impact of discrete and continuous patterns on student problem solving. Suburban middle school students (N = 173) were asked to reason with and across graphical, symbolic, tabular, and verbal representations, including making near and far predictions and producing mathematical generalizations. In addition to varying the representations given to students, one of the critical distinctions of the design was whether the patterns were presented in either a discrete or continuous mode. When student responses were factor analyzed, the authors identified two underlying dimensions to students' thinking: one that favored discrete reasoning (they called it the *instance-based mode*), and one that favored continuous reasoning (the *relational mode*). Each mode of reasoning exhibited different patterns of responses, suggesting that they drew on some independent cognitive processes and had separable developmental trajectories.

There is ample evidence that students draw on reasoning processes that favor discrete and continuous forms of patterns. Which of these two forms of reasoning develops first? Which types of problems do students find easiest? Some prior work speaks to these questions. For example, Carswell (1992) posits that the skills at attending to a specific point develop earlier than those directed at global judgment and integration. Monk (1992) describes how calculus students find questions regarding global qualities of a function that cut across time as far more difficult than point-wise questions. The "point-wise" view reflects students' tendencies to focus on the "level of specific values, of inputs and outputs" rather than "overall patterns of behavior" (Monk, 1992, p. 193). Students do not necessarily make useful connections between discrete forms of representations such as a point-wise graph, and continuous forms such as a line graph of the same function. When given opportunities to reason, students tend to favor discrete representations, and will even assign point-wise interpretations to continuous line graphs (Carswell, 1992; Monk, 1992; Selden & Selden, 1992), sometimes at the cost of obtaining a more global interpretation of a pattern (Goldberg, 1998). Findings like these may be explained, in large part, by the curricular emphasis on making graphs from tables of discrete entries that permeates students' early mathematics education (Driscoll, 1999). Based on these findings, it appears that reasoning about discrete patterns is more accessible to students and may appear earlier in their mathematical development.

Integrating Representations

Establishing connections among representations is a central goal of algebra instruction (Brenner et al., 1997; Cuoco & Curcio, 2001; Driscoll, 1999). Yet, linking between representations is very challenging, and students exhibit many shortcomings in this area (e.g., Knuth, 2000; Swafford & Langrall, 2000). Case's theory predicts that the ability to integrate across representations emerges after developing fluency with the component representations. Based on this, we expect younger students to exhibit the greatest difficulty reasoning with combined representations, and for performance levels to increase with age.

Task Demands

In addition to the design characteristics of how the patterns are presented, it is important to consider the task demands based on the specific pattern generalization questions asked of students. In their review of the literature on graph comprehension, Friel and colleagues (2001) pointed to a "somewhat surprising consensus" (p. 130) across a broad range of articles on the three dominant types of questions asked of students. At the elementary level, the emphasis was on *reading the data* (Curcio, 1987) directly from the graph. At the intermediate level, *reading between the data* and drawing inferences was the focus. In the final level, the emphasis was on *reading beyond the data* (Curcio, 1987) and "reduction of all the data to a single statement or relationship about the data" (Bertin, 1983, in Friel et al., 2001, p. 130). There is evidence that students learn to make abstractions better when they first can formulate predictions for specific instances and then apply inductive reasoning (Koedinger & Anderson, 1998). As Friel and colleagues point out, different task demands elicit different levels of comprehension. Based on this prior work, we can expect that students will perform best at making near predictions, struggle a bit more with making inference-based far predictions, and have the greatest difficulty reducing patterns presented in discrete and continuous forms to a single abstract statement of the underlying quantitative relation.

Summary of the Literature

Graphical and verbal representations of patterns, on their own and in concert, are important for establishing representational fluency. Patterns can also be presented as discrete collections of related instances or as a continuous relationship among varying quantities, and students appear to have access to reasoning processes that parallel these differences (Kupermintz & Nathan, 2004). While much has been studied about these different representational types, there is a need to better understand how students' abilities to reason with them develop over the middle school years. Prior research, such as Case's theory of conceptual development, suggests that students' abilities to operate with verbal and spatial forms on the one hand, and discrete and continuous forms on the other, will develop independently of one another. Findings by Koedinger and colleagues (e.g., Koedinger & Nathan, 2004) suggest that a verbal advantage over graphs is likely to be seen; particularly in light of the challenges exhibited by students working with graphs (e.g., Friel et al., 2001; Leinhart et al., 1990). There is also evidence that discrete modes of reasoning may develop initially and provide early support for understanding and generalizing (Carswell, 1992; Monk, 1992). Later, as facility with the individual representations matures, students can be expected to show fluency with combined representations.

RESEARCH QUESTIONS

While the views given above are consistent with prior research, the developmental implications are somewhat speculative at this point. To advance our understanding of the development of representational fluency, we investigated student performance using cross-sectional and longitudinal analyses.

We first examined cross-sectional performance differences among sixth-, seventh-, and eighth-grade students as they made predictions and mathematical abstractions of simple, contextualized linear functions. We then analyzed longitudinal data from a cohort of students for whom we had annual performance assessments across the entire middle school experience.

We approached these data guided initially by the three general research questions listed above. However, the literature review allowed us to refine some of these questions so they better reflected the empirical and theoretical contributions of earlier investigations. The literature also allowed us to make more confident predictions about student performance differences.

1. How do task demands of reading the data, reading between the data, and reading beyond the data affect students' pattern-generalization performances?

We expected that near prediction (NP) tasks (*reading the data*) would develop early, and be easier, than far predictions (FP; *reading between the data*); and FP would develop before abstract (AB).

2. How do students' pattern-generalization performances differ across the middle school grades when they are presented as discrete and continuous patterns?

We anticipated that students overall, and especially the youngest, would perform better on problems presented in a discrete form that favors an instance-based mode of reasoning, as compared to continuous patterns that favor a holistic and relational mode.

3. How do representational differences (graphs versus words, or graphs and words combined) influence students' pattern-generalization performances?

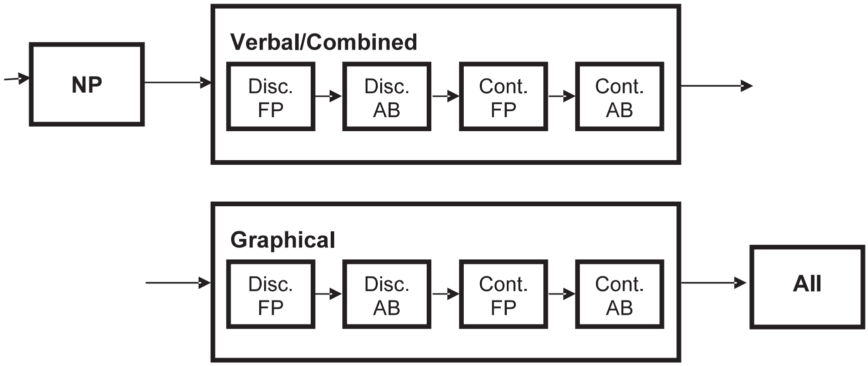

FIGURE 1 Hypothesized developmental trajectory. (NP = near prediction, FP = far prediction, AB = abstraction, Disc = discrete, Cont = continuous).

We expected to see a verbal advantage (Kaput, 1992; Koedinger & Nathan, 2004) of words over graphical representations across the grades, although it is likely to be particularly strong among the youngest students who are likely to have the hardest time with graphs.

We tried to integrate these various predictions into a single developmental model for the typical student taking our assessment. The hypothesized trajectory is illustrated in Figure 1. In considering the interactions among the three controlled factors (task type, representation, and presentation mode), we expected that students would demonstrate success with NP at the earliest stages of pattern-generalization ability, regardless of representation and presentation form. We then expected the advantages of verbal representations (in both the verbal and in combined representations) to become more evident as the pattern-generalization tasks became more demanding, replicating the *verbal advantage* reviewed earlier. We also expected to see an advantage for discrete patterns over continuous ones, in both the verbal and graphical representations. Once these differences were accounted for, we would expect students to succeed on FP tasks prior to AB tasks.

METHOD

Participants

Participants were 372 middle school students (122 sixth-graders, 115 seventh-graders, and 135 eighth-graders) from a middle-class community. The school used the *Connected Mathematics* curriculum (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998). Data collection took place every fall for three years and in the spring of the third year using an algebra assessment that addressed many aspects of algebraic reasoning, including students' interpretations of the equal sign, uses of variables, writing and solving equations, and fluency with graphs and words. Findings on students' thinking and development of equal sign, variables, and equations are presented elsewhere (Alibali et al., this issue; Knuth et al., 2005; Knuth et al., in press; Stephens, 2005). In this article, we focused our analyses on two multipart items from the assessment (described below) that documented students' abilities to predict and generalize linear patterns from discrete and continuous patterns represented with words and graphs. We will first present cross-sectional analyses of the three grades for data collected during the first administration of the assessment. We will then present findings from 81 sixth-graders for whom we have complete longitudinal data through the end of eighth grade.

Materials and Procedures

The assessment presented pattern-generalization problems in the form Y = mX + b, where b and m were small, positive, whole numbers. Problem representations varied by test form between students, and were given either as graphs, words, or a combination of the two (for the combined version, see Figure 2). Each of these test forms contained two problems (a within-subject factor)—one presenting the pattern as a collection of discrete instances (Figure 2a), and one presenting the pattern

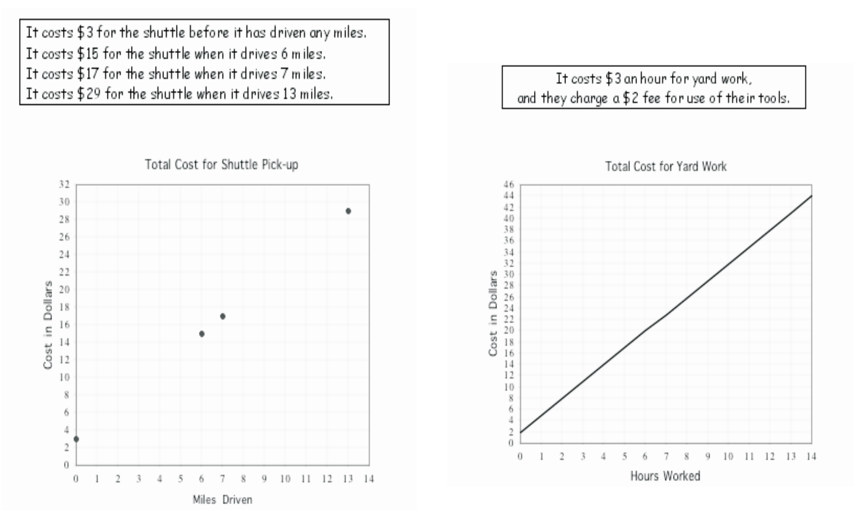

FIGURE 2 Examples of two pattern-generalization problems using combined graphical and verbal representations presented in (a) a discrete mode (a verbal list or point-wise graph) on the left, or (b) a continuous (verbal rule or line graph) mode. The verbal condition received only the text boxes (top), the graph condition received only the graphs (bottom), and the combined condition received both the text and a graph, as presented.

in a continuous form (Figure 2b). In addition, each problem consisted of three parts: an NP, an FP, and an AB task. Examples of the two problems in graph form are shown in Figure 3. Thus, there were a total of six items to be solved by each student.

NP items asked for a Y-value corresponding to a given value of X (e.g., X = 10). It is a near prediction because the solution was within the range of given information when the item was presented in its discrete form. For example, in Figure 3 we see in part a of the problem that X = 10 is asked about for a graph that reaches 14. For FP tasks, students had to give a Y-value for a given value of X (e.g., X = 31) that was outside the range of given information when that item was presented in its discrete form, and therefore required extrapolation beyond the graph or list of entries. Finally, for AB tasks, students were asked to "Write a mathematical equation that you could use to find" the specific X-Y relation mentioned in the problems, such as the total cost to make any number of copies of a CD (Y) if you knew the number of copies you wanted (X). The AB item addresses students' abilities to reduce "all the data to a single statement or relationship about the data" (Bertin, 1983, in Friel et al., 2001, p. 130). The NP, FP, and AB tasks were presented in the same order on every test booklet for both discrete and continuous patterns.

Students were randomly assigned to one of three representational forms (graphical, verbal, or combined) and were given the same assessment form across the four test administrations. The assessment was given during students' regular mathematics classes and administered by the teacher, following prepared instructions. Students were told to use a pen, show their work ("If you want to cross something out, just draw a line through it"), and to not use a calculator.

Scoring of Task Performance

Students' written responses were scored for each of the six items; they received 0 points when an answer to an item was incorrect, and 1 point when an answer was correct. We used strict criteria for NP and FP tasks, meaning the responses had to exactly match the computationally correct answers. Each student was assigned a score ranging from 0 to 3 for each of the three multipart items. They could receive up to 3 points each for the two discrete and continuous pattern problems, for a total possible score of 6 points. A second coder rescored a randomly selected 20% of the data to establish reliability. Interscorer reliability was 99%.

Coding of Problem-Solving Strategies

Students' problem-solving strategies were also coded for each item. Prior studies (e.g., Hall, Kibler, Wenger, & Truxaw, 1989; Nathan & Koedinger, 2000) show that aspects of student thought can be determined from careful analyses of solution protocols. Only those strategies that were relevant to the students' answers were coded.

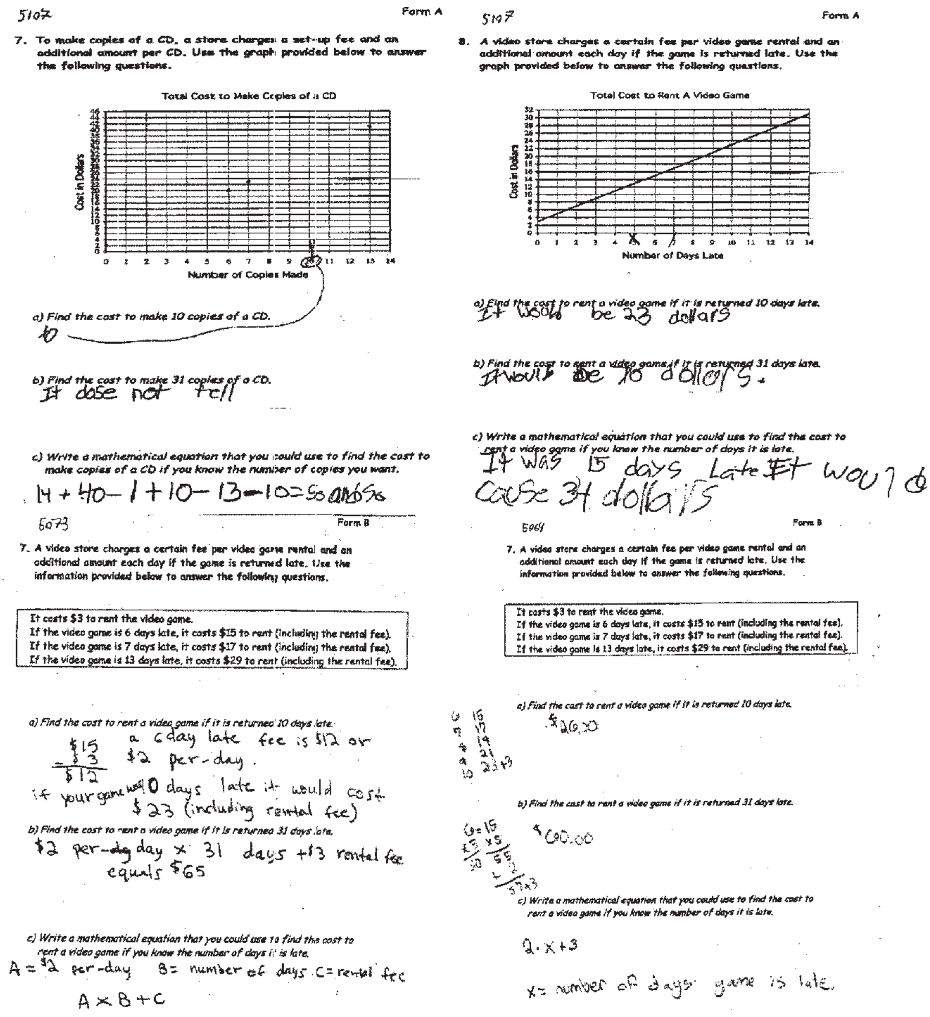

FIGURE 3 Example strategies from a sixth-grader, shown (a) using the reading the graph (RG) for discrete and (b) continuous presentation modes, (c) arithmetic rate (ARR), and (d) linear combination (LC).

Reliability for students' strategies and errors was established by a second coder for a randomly selected 20% of the data. The agreement between coders was 90%.

We will briefly describe several of the strategies that proved to be important for our findings. Answer-Only (AO) responses included no work, but showed a response (unlike No Response (NR), which was blank). Use of AO suggests that these items did not place a large load on students' limited working memory capacity, perhaps because they were seen as computationally easy, solved by direct perception, or students were merely guessing. Reading the Graph (RG) was evident in both discrete (Figure 3a) and continuous (Figure 3b) patterns when students made marks or traced lines along any portions of the graphs. Arithmetic Rate (ARR) is an arithmetic-based strategy in which students "plug in" values for the rate and intercept. It is often taught explicitly by middle school teachers. In the example (Figure 3c) the student first used the unwinding strategy (cf., Koedinger & Nathan, 2004) to strip away the $3 overhead rental charge from the $15 late fee that was assessed for 6 days. The student then computed the rate of "$2 per day," which was applied to 10 days, and then added to the $3 rental fee, for an answer of $23. Notice that in part c of the problem, this seems to support a general relation, $A \times B + C$. With the Linear Combination (LC) strategy (Figure 3d), the student uses the established correspondence between X (days late) and Y (dollars charged) to set up a general linear relation, much like proportional reasoning, but one that includes the intercept or overhead amount. Notice in part b, the list entry for 6 days late is literally equated with the value of 15 (the total late fee). Since the student's goal is to compute the total late fee for 31 days, the student sets up a relationship between 6 and 31; that is, 31 days = 6 days $\times$ 5 + 1 day. To determine the parallel relation in terms of dollars (late fee), the student multiplies 15 (i.e., the fee for 6 days) $\times$ 5 + 2 (the fee for 1 day).

RESULTS

Our aim is to articulate aspects of middle school students' development of representational fluency for patterns using cross-sectional ($N = 372$) and longitudinal performance data ($N = 81$) and insights from students' strategy use.

Cross-Sectional Analyses

We will first consider grade-level differences for overall performance of the students during the first year of assessment, in which students can score a total of 6 points for the three tasks (NP, FP, AB) presented in both discrete and continuous modes. As one might expect, there were reliable differences across the grades for overall performance, $F(2, 369) = 21.32$, $p < .05$. During the first year of the study, eighth-graders ($M = 2.9$ out of 6 total points) significantly outperformed seventh-graders ($M = 2.4$), and seventh-graders significantly outperformed sixth-graders ($M = 1.4$), with the greatest difference in representational fluency occurring between sixth- and seventh-graders.

Differences Between Prediction and Abstraction. Prediction tasks are typically seen as precursors to abstraction (e.g., Koedinger & Anderson, 1998). It was our expectation (Figure 2) that NP and FP tasks, which require students to find a corresponding output for a given input, would generally be solved more easily and earlier than AB tasks, which involved reformulating the underlying relationship into an equation. It was also expected that NP tasks would be easier than FP tasks for all students, but especially the youngest in our sample. NP tasks asked for the dependent variable (Y) for a given input amount (X), for a small input. This emphasized *reading* in the graphical case, since both the given and unknown amounts were within the range of values shown. For the FP tasks, students had to reason outside the given range.

Consistent with the literature and with our expectations, NP tasks were significantly easier ($\alpha = .05$) than either FP or AB tasks. This performance difference was apparent in all comparisons, including grade, representational format, and the mode of pattern presentation.

FP tasks were more readily solved than AB by sixth-graders. However, the success rates for FP and AB tasks were quite similar among seventh-graders (30% correct for FP, 34.3% for AB). By eighth grade, somewhat surprisingly, students were more likely to get AB problems correct than FP items (46.3% correct for AB, 34.8% for FP).

These results raise the question about the relation of one's FP strategy to one's success at the AB task. This is of interest, since the FP task asked students to extend their reasoning beyond the given information. Normatively, we expect that solving FP will lead students to develop an abstraction to support the generalization needed to make a far prediction, as suggested by the hypothesized developmental trajectory of Figure 2. That is, we expect there is some common underlying process that allows one to make far predictions and also mediates performance on the AB task. If this is so, performance on FP and AB tasks should be statistically related.

Specifically, we wanted to know if students maintained a consistent pattern of performance on FP tasks relative to AB tasks; that is, were students always better on FP than AB, or were they sometimes better at FP and sometimes better at AB? Since performance on these two items is paired (each student was given both) and nominal (i.e., they can only get each correct or incorrect), we used the McNemar nonparametric test. If the test is significant, then it is assumed that the performance on the two items is so inconsistent as to warrant two different processes involved in the reasoning.

We found no differences for sixth- and seventh-graders for either discrete or continuous patterns. This close relation of FP and AB performance suggests that sixth- and seventh-graders are using similar abstraction processes when reasoning about FP and AB tasks. However, significant McNemar tests of the eighth-grade data indicated AB tasks were easier than FP tasks in both the discrete ($p = .01$) and continuous ($p < .001$) cases. This could be construed as an important transition point in mathematical development in eighth grade, where students' reasoning about pattern abstraction deviates from, and in this case transcends, the reasoning processes involved in making FPs.

Presentation Mode. We next consider performance differences due to the within-subjects manipulation of the mode of presentation. Our hypothesis was that students would show an advantage for discrete patterns because they support direct inspection of and calculation on instances (Figure 2a). This advantage for exemplars (Monk, 1992) is expected to be particularly strong for the youngest students, who have not had algebra instruction.

As the data show (Figure 4), the hypothesis favoring discrete presentation was not upheld. In fact, students exhibited an advantage for continuous patterns overall, $F(1,363) = 21.48$, $p < 0.001$. Specific grade-level tests showed that the advantage for continuous patterns holds for sixth and eighth grade, but not for seventh grade. For sixth-graders, the advantage translates to a nearly 50% increase in performance (out of 3 total points). In eighth grade, the advantage is nearly 40%.

The continuous pattern presents the linear function in a concise, relational form. Evidently this is more accessible to students and more supportive of pattern generalization than one that provides students with discrete instances that fit the pattern. We will next look to the students' solution strategies to better understand these results.

As Table 1 shows, students use a variety of strategies to solve the prediction problems. However, several strategies—particularly AO, ARR, and RG—were

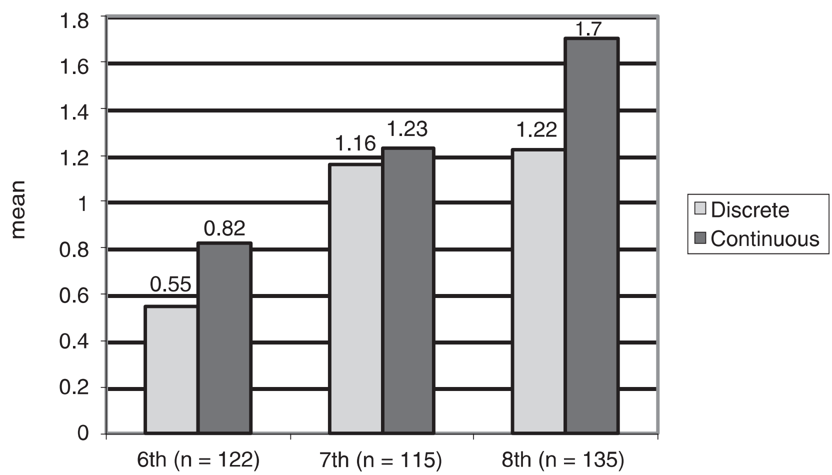

FIGURE 4 Analysis of cross-sectional data by presentation mode (within subjects, $N = 372$) shows a performance advantage for continuous patterns. Differences are significant for sixth- and eighth-graders. Totals are out of 3 points.

TABLE 1
Frequency of Strategy Use for Solving NP and FP Tasks in the Cross-Sectional Data (N = 372)

	AG	*ARR*	*RA*	*AO*	*RG*	*ARO*	*MULT*	*LC*	*NR*	*DK*	*O*	*U*	*G*	*NP*	*Total*
6th Grade															
NP discrete															
Correct		4	7	20	1	1	2								35
Incorrect	1		8	30	12				28	3	2	1	1	1	87
FP discrete															
Correct		8	2	5		1		1							17
Incorrect		1	3	45		3		8	32	7	2		2	2	105
NP continuous															
Correct		9	1	39	3										52
Incorrect		3	5	33	2	2			22		1	2			70
FP continuous															
Correct		12	1	11											24
Incorrect		7	3	38		5		6	32	2		2	1	2	98
7th Grade															
NP discrete															
Correct	3	5	5	34	6	1	1						1		56
Incorrect		2	3	23	5	2			19	4		1			59
FP discrete															
Correct	2	12		22		1									37
Incorrect		5	2	26		4		5	30	6					78
NP continuous															
Correct		10	1	52	6										69
Incorrect		3		20					20	2		1			46
FP continuous															
Correct		12		21											33
Incorrect		10		35		1		5	23	7		1			82

(continued)

TABLE 1 *(Continued)*

	AG	*ARR*	*RA*	*AO*	*RG*	*ARO*	*MULT*	*LC*	*NR*	*DK*	*O*	*U*	*G*	*NP*	*Total*
8th Grade															
NP discrete															
Correct	2	13	25	22	11	2									75
Incorrect		10	10	20	9	5	1		4	1		1			60
FP discrete															
Correct	1	25		11				1							38
Incorrect		17	4	36		11		15	11	1		1	1		97
NP continuous															
Correct	1	33	3	57	5										99
Incorrect		4		20	2	2			8						36
FP continuous															
Correct		38	1	14	1	1		1							56
Incorrect		14	1	31		5		9	18	1					79

Note. AG = Algebraic; ARR = Arithmetic rate; RA = Recursive addition; AO = Answer only; RG = Reading the graph; ARO = Arithmetic other; MULT = Multiple strategies; LC = Linear combination; NR = no response given (left blank); DK = “Don’t know”; O = Other; U = Unknown; NP = “Not possible.”

more successful when applied to continuous patterns than when used with discrete patterns. Strategy use was clearly dominated by AO responses. Notably, AO use on continuous items had a much greater likelihood of success (41%) than discrete patterns (25%). Similarly, we observed greater success with the ARR strategy when applied to continuous patterns (50%) than discrete patterns (28%).

RG was also more reliable when continuous patterns were presented. Figure 3 shows an important contrast between students' uses of the graphs to make predictions when in the discrete or continuous modes. Both sides of this figure show work from the same sixth-grade student. As noted on the left-hand side of the figure, when the pattern was presented as a point-wise graph, the student used the value on the *X*-axis to respond to the NP question (*part a*), and gave an answer of 10. While this is not correct, it demonstrates how the graph afforded access to this item. However, when asked to reason about an FP value (*part b*), the student saw the graph as fundamentally bounded by its frame (cf., Bieda & Nathan, 2006) and responded "It dose [sic] not tell." In contrast, the same student used the continuous line graph on the right-hand side of the figure to read off the answer to the NP problem, and attempted to reason about the FP problem (*part b*). Although the student's answer was incorrect, this is not due to a perceived constraint of the graphical frame.

As noted above, AB items required a different form of response than NP and FP items—students had to give a proper mathematical expression or equation that would explain the underlying pattern. While the expression or equation could contain letters and words, it could not be exclusively in verbal form. It also had to express a generalized relation and could not simply describe the procedure for a given instance. This latter point is significant, since one of the most common errors was to find the value of an instance, rather than a generalized solution. Sixth-graders were, understandably, far more likely to solve for an instance than older students. However, seventh- and eighth-grade students were more likely to take this approach when they were given a discrete pattern. All of these strategy differences contributed to the overall advantage for continuous over discrete patterns.

Representation Differences. As a between-subjects manipulation, students solved problems in graphical (n = 125), verbal form (n = 124), or with both representations combined (n = 123). We first considered differences between graphical and verbal representations. We expected to observe a verbal advantage, paralleling previous studies of reasoning with algebra equations. The argument has been advanced that verbal representations tap into the comprehension processes students have already developed throughout their early years of learning language, reading, and interacting socially, and this allows for greater sense-making, more self-directed error correction, and the opportunity to employ weak methods to

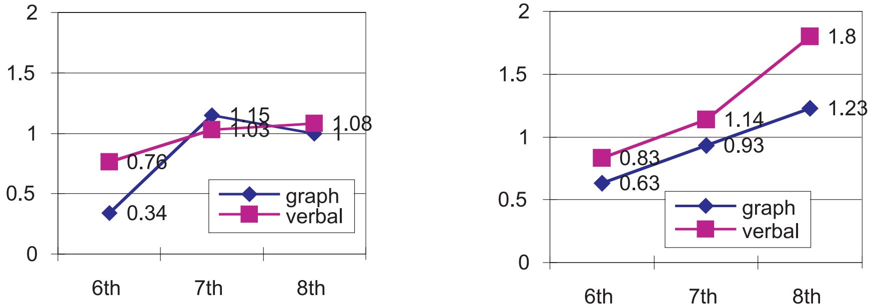

FIGURE 5 In the cross-sectional analysis ($N = 372$) there is no verbal advantage for items that present patterns in (a) the discrete mode, but there is in (b) the continuous mode. Totals are out of 3 points.

invent appropriate and robust solution strategies on the fly (Koedinger, Alibali, & Nathan, in press; Koedinger & Nathan, 2004).

In the current data set we found the expected verbal advantage. Pattern-generalization performance across items and grades was higher when those patterns were presented in words as compared to graphs, $F(1,243) = 5.55$, $p < .05$. This replicates prior findings that show an advantage over equations with high school and college students. However, as Figure 5 suggests, this advantage is not homogenous, but specific to continuous patterns (Figure 5b). A post-hoc test confirms this, $t(243) = 2.36$, $p < .05$.

In addition to an overall verbal advantage, there was an advantage for discrete patterns presented verbally for the youngest students in the study (Figure 5a; $F(1, 81) = 4.32$, $p < .05$), suggesting that those with the least mathematics training made the most from the verbal representations; alternatively, these young students also had the most difficulty with graphs.

Our design also allowed us to observe whether there was added value for pattern generalization when graphical and verbal information was combined. This question is of interest because students are expected to eventually develop fluency with words and graphs that would allow them to make use of the synergy provided by these two, complimentary representations (Case, 1985; 1992; 1996; Kupermintz & Nathan, 2004). Our results show an overall main effect for representation, with an advantage for combined representations, $F(1,363) = 10.37$, $p < .05$. While the verbal representation ($M = 1.11$ correct items on average) is superior to graphs ($M = .88$), the combined use of graphs and words produced the highest performance overall ($M = 1.34$). Individual contrasts show combining graphs and words were superior to graph and verbal representations alone ($p < .05$). Thus, the added understanding available to students by combining words with graphs is particularly useful for their pattern-generalization reasoning.

Summary of Cross-Sectional Analyses

Performance naturally followed grade-level differences. We also found the expected ordering among task difficulty, with NP tasks easier than FP, which were in turn easier than AB (data reduction and reformulation). This is consistent with our notions that each level of generalization builds somewhat cumulatively on the prior one. However, one unexpected finding was that eighth-graders exhibited a transition point in their mathematical development in which their abilities to abstract from patterns were better than their abilities to make predictions. This appeared to be somewhat robust because it was evident from the data on both the discrete and continuous patterns. Another surprising finding was that performance on problems with continuous patterns was consistently higher than problems with discrete presentation of the same underlying functions. Students executed the most commonly used strategies (AO, ARR, RG) successfully when reasoning about the continuous patterns. Consistent with our predictions, there was a general verbal advantage over graphs. While this advantage was evident among sixth-graders for both discrete and continuous patterns, we failed to predict that the advantage only applied to discrete patterns for older students, although it widened considerably by eighth grade. Finally, students who received combined representations outperformed those given the individual representations, proving an unexpected synergy even at the earliest grades. Several predictions were overturned, principally, the greater success with continuous patterns, its apparent relation to the verbal advantage, and early advantages for combined representations. This provides new information for the emerging developmental trajectory as we delve into the longitudinal data.

Longitudinal Analyses

In the longitudinal data set ($N = 81$), we explored how grade-level differences in assessment performance and strategy use changed over the middle school years with increasing instruction and mathematical maturity. Our analysis of overall performance (a total of 6 points) across the two presentation modes for the three tasks (NP, FP, AB), showed students exhibited general improvement, $F(3, 240) = 15.55$, $p < .01$. Specifically, students in the longitudinal cohort showed a reliable gain from sixth grade (27% correct) to seventh grade (43%) performance, and a significant gain during eighth grade from the fall (41.7%) to the spring (50%). However, students showed no reliable gains or losses in the time between the seventh and eighth grade fall assessments. Since this is a somewhat puzzling result, a number of post-hoc analyses were performed on class membership, teacher effects, and comparisons at different achievement levels to determine whether there were averaging masked performance gains for any subgroup of the cohort, but these all proved to be insufficient to explain the flat performance changes between the second- and the third-test administrations. Still, the picture of student change shows a

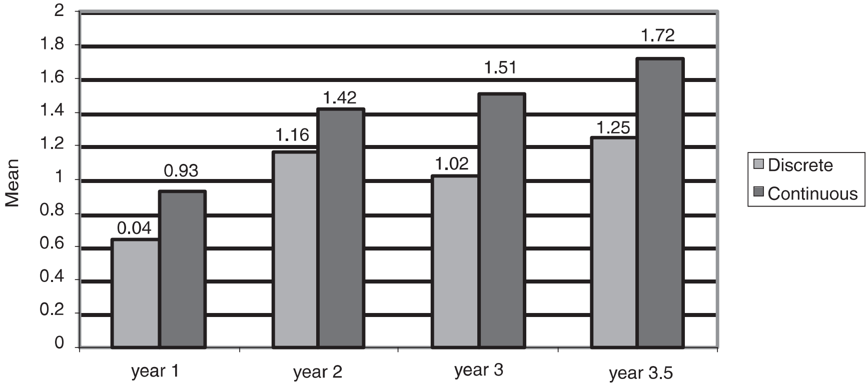

FIGURE 6 An advantage for continuous patterns is evident in every assessment administration given to the longitudinal cohort (N = 81) during the three-year study. Totals are out of 3 points.

near doubling (87.5% increase) in overall performance in pattern generalization from the beginning of sixth to the end of eighth grade, with most (over 70%) of the increase occurring during the transition from sixth to seventh grade.

Presentation Mode. Comparisons between discrete and continuous modes of pattern presentation showed that, contrary to our initial hypotheses (but consistent with the cross-sectional data) there was a reliable advantage for continuous patterns in both graphical and verbal representations (line graphs and verbal rules), $F(1,78) = 24.2, p < .01$. As shown in Figure 6, the advantage for continuous pattern was present in the earliest years of this study and persisted throughout the middle school years, with a significant advantage for continuous patterns found at each of the four test administrations. As students matured, the advantage for continuous patterns grew larger.

Representation Differences. Students in the longitudinal cohort were also asked to reason about patterns that were represented in either graphical (n = 27) or verbal form (n = 28), or with both representations combined (n = 26). As expected, there was a verbal advantage for this cohort. However, as with the cross-sectional data, this was due largely to the advantage of verbal representations over graphs when the patterns were continuous, $F(1, 53) = 4.04$, $p = .05$ (see Figure 7b). While no verbal advantage was evident among the discrete patterns overall (Figure 7a), a post-hoc test revealed a verbal advantage for instances during the second test administration (when the cohort was in seventh grade), $F(1, 53) = 13.6$, $p < .05$.

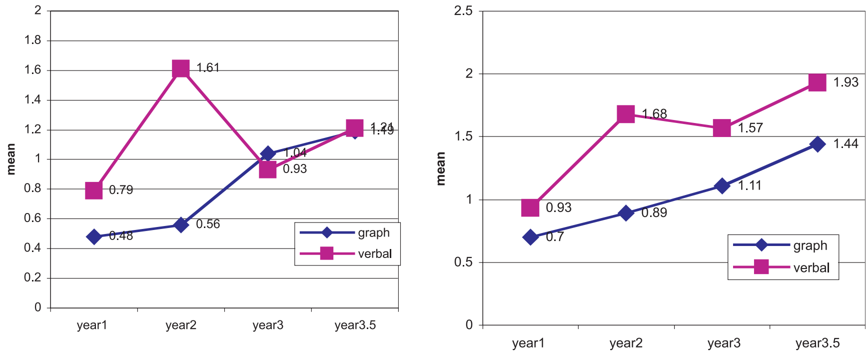

FIGURE 7 Verbal advantage among the longitudinal cohort ($N = 81$) (a) is not found in the discrete mode (except in Year 2) but (b) evident in the continuous mode. Totals are out of 3 points.

In addition, there is an interaction of representation with time ($F(3, 159) = 5.1$, $p < .05$), showing how performance with discrete verbal patterns dropped from seventh to early eighth grade, although it buoyed back in spring, leading to a three-year gain overall. There were no additional interactions with representation, which means that the nature of the verbal advantage is consistent across the four assessment time points.

Developmental Mapping. For the last analysis, we provide a developmental mapping intended to give insights into students' general progression of pattern generalization capability reflected in the cross-sectional and longitudinal data. Recall that students were given the six items either in graphical, verbal, or combined representations. Consequently, we cannot make direct claims about developmental transitions between these representations. We rely instead on the between-group analyses, which suggested that combined representations are most accessible to students, followed by verbal and graphical representations. While the data show a reliable advantage of verbal representations over graphical ones, combined representations show that adding graphical information reliably provided an additional advantage. Consequently, we use these findings to argue that students will have the easiest access to combined representations, followed by verbal and graphical representations. This is reflected in changes made to the proposed developmental trajectory (Figure 8), where within each major box (continuous and discrete) performance proceeded from fluency with combined representations to verbal and then graphs.

Among the repeated measures, we noted that NP tasks were the most accessible to students, regardless of grade level, representation, or mode of pattern presentation. Consequently, this appears to be the natural entry point for members of our

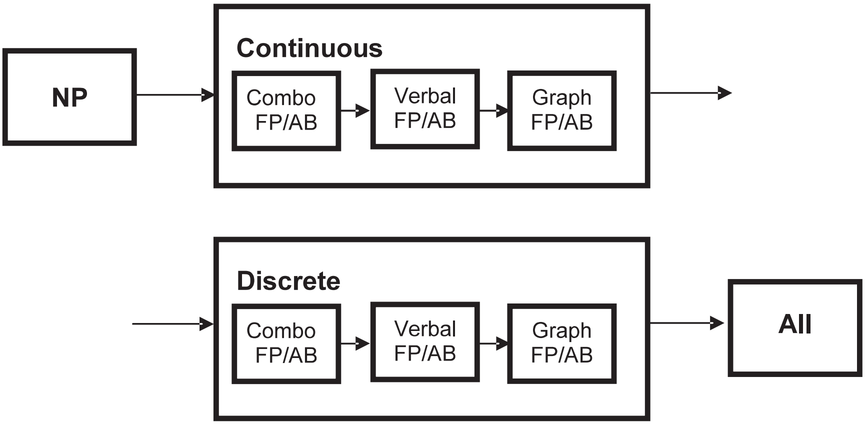

FIGURE 8 Observed developmental trajectory. (NP = near prediction, FP = far prediction, AB = abstraction, Combo = combined representation).

cohort, as well as students across the middle grades. FP and AB tasks depended more on students' pattern-specific reasoning abilities, with continuous patterns more accessible than discrete patterns. Finally, these understandings come together and support students in all the test items. These transitions are summarized in Figure 8. We will explore the implications of this model for research and instruction in the next section.

DISCUSSION

This study set out to explore middle school students' initial and developing forms of pattern generalization, and to propose, and then revise, a hypothetical trajectory for the development of representational fluency using discrete and continuous patterns presented in graphs and words. Previous theoretical and empirical research in the development of mathematical reasoning has proposed there is an inherent dualism between discrete and continuous modes that ultimately lead to an integrated conceptual structure (e.g., Case, 1985; 1992; 1996). Generally, continuous, and especially verbal continuous, presentations of patterns tend to support more reliable execution of solution strategies, and later generalization of the algorithms to algebraic forms (Carpenter, Franke, & Levi, 2003).

Perceptual processes also appeared to impact students' reasoning, particularly with graphs. While this has been documented with graphs (Bieda & Nathan, 2006; Friel et al., 2001) it is also evident in other mathematical formalisms, such as equations (e.g., Landy & Goldstone, in press). Although verbal processes obviously play a significant role in seeing and reading the visual aspects of any representations, there are also ways that distal, cognitive processes are impacted by the spatial configuration of representations. For example, in their analysis of eye-tracking protocols, findings by Kim, Kim, & Kim (2001; Kim & Kim, 2002) suggest that continuous line graphs are more amenable to visual chunking of the data. This holistic quality (e.g., Kupermintz & Nathan, 2004) may facilitate perceptual processes involved in pattern analysis and prediction, even for the simplest NP tasks. Zacks and Tversky (1999) showed college students were more likely to consider trends when given line graphs, but more likely to make contrasts when shown discrete values (bar graphs) of the same data. Recently, Nathan and Bieda (2006; Bieda & Nathan, 2006) showed how some middle school students indicated a "bounded interpretation" of the graphs through gesture and speech, much like the view expressed in Figure 3a. Bounded gesture use tended to predict students' ability to solve FP tasks. In the current study, we found students were more likely to make accurate generalizations in the AB problems when continuous patterns were presented (Figure 6), while discrete patterns led students to conceptualize the requests for abstractions as questions about finding an acceptable instance. Students' reasoning, then, appears to "import" aspects of the working environment and operate on mental representations as though they had the limits of physical (perceptual) phenomena. The influence of the spatial features of these representations reinforces the body- and perception-based nature of reasoning, as suggested by the Embodied Cognition community of researchers (e.g., Lakoff & Nunez, 2000; Wilson, 2002).

We also found support for the predicted verbal advantage. It is quite possible that natural language is most easily understood by students, paralleling historical development (e.g., Kaput, 1992), and that students' verbal comprehension and meaning-making processes, once activated, moderate the selection and execution of solution methods. Verbal representations enjoy a connection to students' larger social and cognitive development (Koedinger & Nathan, 2004; Vygotsky, 1985), and as such, have the potential to play a formative role in the development of one's mathematical reasoning, even while transitioning to the use of formal representations and specialized notation (Gee, 2004).

In previous research (e.g., Nathan et al., 2002), verbal representations of patterns have shown an advantage over other representational forms in areas of algebraic reasoning. The current data replicate this finding, but reveal it to be somewhat restricted to the continuous mode of presentation. We interpret this to mean the verbal advantage is robust across a wide range of student populations, but is also limited to specific types of mathematical reasoning tasks. In reviewing prior work on this matter, it is apparent that the story and word equation problems used were primarily of the continuous kind (See, e.g., Koedinger et al., in press; Koedinger & Nathan, 2004). Story problems present the basic quantitative relations using the structural and episodic properties of the referent situation. Word equations use concise language to describe the quantitative structure relationally or procedurally rather than as a collection of exemplars. Thus, the present findings do not appear to contradict earlier results showing a verbal advantage for students in other areas of algebra, but help clarify the nature of the advantage. Since this is the first study to specifically manipulate the presentation mode and the representation, it has made a latent factor salient and exposed the manner in which the verbal advantage operates.

In addition to refining our understanding of the verbal advantage, we found a surprising relation between FP and AB tasks. A normative view (see Figure 1, as well as Bertin, 1983; Curcio, 1987; and Friel et al., 2001) assumes that FP is a necessary precursor to articulating an abstract relation. However, for our eighth-graders, AB appeared to operate independently from predicting instances. This suggests that there may aspects of the patterns themselves, rather than generalizations of calculation procedures applied to the patterns, which can directly mediate the process of abstraction. This suggests that abstraction tasks could be considered as aims unto themselves rather than as scaffolded exclusively by FPs. The prerequisite, however, seems to be that other, more general mathematical maturity must be in place for these independent AB processes to be enabled.

IMPLICATIONS FOR MATHEMATICS LEARNING AND INSTRUCTION

The results reported here provide some new insights about students' uses of formal representations when asked to reason about patterns. The findings have implications for any investigations within the physical and social sciences in which students are asked to make predictions and draw generalizations from limited information. Specifically, our data suggest that once one moves beyond basic NP tasks that emphasize reading the data, we can expect continuous patterns to provide greater support for making predictions and generalizations than was previously suggested. We also presented evidence that presentation of continuous patterns serve as natural entry points for algebra learners seeking to represent the quantitative relations in symbolic form. When continuous patterns are presented verbally, they are especially accessible, enabling *younger* students to perform many of the same solution methods more successfully. Success with continuous forms also tended to lead to more advanced development later on, as those students moved further along developmental trajectories and succeed on items (such as AB and FP tasks) that were solved by relatively few students overall.

The findings here enjoy some corroboration (e.g., Kupermintz & Nathan, 2004; Nathan et al., 2002). As such, they contribute important guidelines for designing future curricula and for guiding teachers during lesson planning and assessment design. We believe that, above all, these findings reveal the importance of adopting a developmental perspective that considers the entry points and areas of attainment seen among students through the middle school years. This central principle can be used to organize many facets of instruction and research into mathematical reasoning.

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