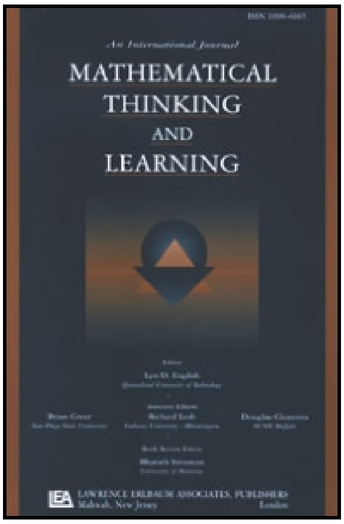

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A Framework for Understanding and Cultivating the Transition from Arithmetic to Algebraic Reasoning

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INTRODUCTION

A Framework for Understanding and Cultivating the Transition from Arithmetic to Algebraic Reasoning

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As organizers of this special issue and investigators in the Supporting the Transition from Arithmetic to Algebraic Reasoning (STAAR) project, we both felt we would be remiss if we did not acknowledge the singular importance of Jim Kaput's influence on this body of work. From the earliest meetings of the project team, Jim's deep thinking about the nature of algebra and of algebraic reasoning and instruction served as a guide. Jim contributed substantially to each of the tiers. But, more significantly, Jim appreciated the need and challenge to address each of these areas of algebra education research in a systemic manner. He agreed unflinchingly when asked to join our advisory board, and the project benefited immensely from his writings and from various discussions on the work. It is a patent understatement that his early death is a loss for the mathematics education community in general and for the community of algebra researchers in particular. Yet we are proud to say his ideas and contributions will continue in projects such as this. Therefore, it is only fitting that we dedicate this set of papers to Professor James J. Kaput.

Algebraic reasoning stands as a formidable gatekeeper for students in their efforts to progress in mathematics and science, and to obtain economic opportunities (Ladson-Billings, 1998; RAND, 2003). Currently, mathematics education re-

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search has focused on algebra in order to provide access and opportunities for more students. There is now a growing awareness that the essential concepts that make up school algebra are accessible to students before secondary-level education, and that earlier introduction could facilitate students' algebraic development (Carpenter, Franke, & Levi, 2003; Kaput, Carraher, & Blanton, 2007; National Council of Teachers of Mathematics [NCTM], 2000, National Research Council [NRC], 1998; RAND, 2003). In order to understand middle school students' transition from arithmetic to algebraic reasoning, and to develop and evaluate effective educational approaches to improve the learning and teaching of increasingly complex mathematics, future efforts need to be grounded in sound theory. This theory needs to encapsulate both how students develop algebraic reasoning and acquire domain knowledge, and the beliefs, knowledge, and existing practices of teachers. The theory must also acknowledge the complexity of this area of study, including its multi-tiered nature, diversity of settings and participants, and the high degree of interconnectedness among important components. For example, to understand students' algebraic reasoning and development, we need to pay attention to classroom interactions, student preconceptions, teachers' beliefs about mathematics and learning, how teachers' beliefs and instructional practices shape the learning environment, and how teachers themselves learn and change.

In an effort to conduct research along these lines, a team of researchers from the University of Colorado, University of Wisconsin, and Carnegie Mellon University developed a framework that guided a recent IERI-funded project,[1] Supporting the Transition from Arithmetic to Algebraic Reasoning (STAAR). This framework outlines a comprehensive, systemic research and development program to address several inter-related areas, or *tiers*, that we see as central to this effort—student learning and development, teacher beliefs, knowledge and practice, and professional development (cf., Lesh & Kelly, 2000). Figure 1 depicts the integrative arrangement of the three tiers to allow research and development activities to be positioned within the system. Our approach emphasizes the parallel structures and processes among these tiers, viewing them as distinct but inseparable aspects of a unified system.

The authors of the papers in this volume conducted research and development within this multi-tiered and dynamic framework in an attempt to move beyond piecemeal, disconnected insights to reach a deeper appreciation of the conceptual terrain and learning processes to inform instruction, curriculum development and professional development. The foundation on which this multi-tiered structure operates gives a sense of the domain of algebra as it is learned in schools.

[1]The Interagency Education Research Initiative (IERI) is a federal partnership that includes the US Department of Education-Institute of Education Sciences (IES), the National Institute of Child Health and Human Development (NICHD), and the National Science Foundation (NSF).

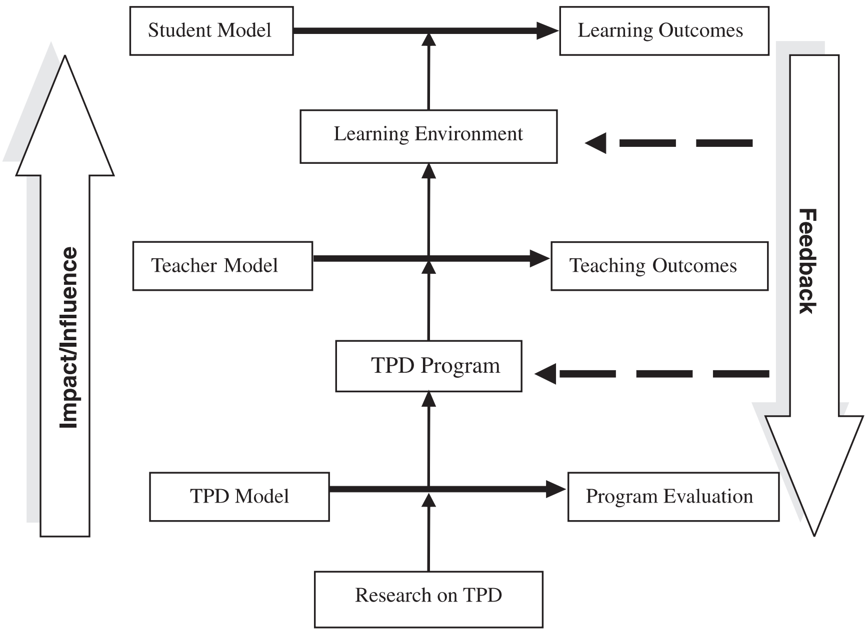

FIGURE 1 Diagram showing the three distinct but interrelated tiers that form the basis of our conceptual framework: (1) student learning and development (top); (2) teacher knowledge, beliefs, and practices (middle); and (3) teacher professional development.

THE DOMAIN OF ALGEBRA AS THE LEARNING TERRAIN

James Kaput considered algebra the keystone of mathematics reform because of its mediating role between arithmetic in the primary grades and calculus and functions in high school and beyond. Kaput's (1998) framework provides a useful organization of algebraic activities and skills to guide investigation into core components (representations, strategies, and knowledge structures) and processes (argumentation, modeling, generalization, and problem solving) of algebraic activities. Chazan (2000) also notes several concepts that are introduced in algebra: (a) a relational or structural conception of the equal sign (cf., Kaput's second aspect), which contrasts with the operational conception favored in arithmetic (Rittle-Johnson & Alibali, 1999); (b) literal symbols as variables that can denote a *set* of values (Kaput's first form), rather than just a single value; and (c) the trade-offs between symbolic, tabular, graphical, symbolic, and verbal representational forms (Knuth, 2000). It is within this context and several recent summative reports (e.g., NRC, 2005; RAND, 2003), that we aligned our research around two *core concepts*—equality and variable; and three *major aspects* of algebraic reasoning—the use of formal representations and representational fluency, pattern generalization and function, and problem solving.

What, then, does it mean to reason algebraically? How can this reasoning be cultivated in middle school classrooms? To address these questions, we turn to the theoretical underpinnings, and relevant empirical findings within each of the three tiers.

TIER 1: STUDENT ALGEBRAIC REASONING

There is a growing recognition that "[e]ffective instruction begins with what learners bring to the setting" (Bransford, Brown, & Cocking, 1999, p. xvi). Students contribute powerful general and mathematics-specific ideas, knowledge, and skills to the classroom environment (Carpenter, Franke, & Levi, 2003). These nascent forms of mathematical reasoning are ingrained in internal mental representations and dispositions, but also in socially determined patterns of participation, within and outside school. Prior knowledge and conceptions, both formal and informal, also play an important role in student performance and later development. If we are to identify inroads into students' learning and understand obstacles to development, we must understand students' prior knowledge.

Students may choose not to use formal approaches in reasoning about mathematical problems even when they know them (Koedinger & Nathan, 2004). Students also miss fruitful connections. For example, Knuth (2000) showed that advanced high school algebra students do not readily connect graphical representations such as the Cartesian coordinate system to their knowledge of equations, and fail to use graphs even when graphical solutions are easier and more efficient. Yet young students do exhibit algebraic reasoning, even before instruction (Carpenter et al., 2003). Nathan and Koedinger (2000a) showed that algebraic symbolisms could be introduced in sixth-grade classes as natural generalizations of students' invented strategies. Swafford and Langrall (2000) found that sixth-graders could use algebraic equations to describe and represent generalizable problem situations even prior to formal instruction, but they rarely used equations to solve related problems.

Studies by Koedinger, Nathan, and Alibali (in press; Koedinger & Nathan, 2004; Nathan & Koedinger, 2000b) showed that students performed better on *verbally presented* story and word-equation problems than on matched symbolic equations. Verbal problems were also more likely to elicit invented strategies such as "guess-and-test" and "unwind" (a working backwards strategy; Kieran, 1988; Nathan & Koedinger, 2000b). Further, these invented strategies were more effective than formal approaches—even for students who had completed over a year of formal algebra instruction. Verbal representations appear to mediate the generation of successful strategies by allowing students to carry out quantitative reasoning in a familiar verbal problem form. Further analyses of high school students' problem-solving performances (Nathan & Koedinger, 2000b) suggest that students tend to follow a *verbal precedence model,* in which verbal reasoning about quantitative relations precedes symbolic reasoning (see also Case & Okamoto, 1996; Kalchman & Case, 1998).

However, verbal strategies are not sufficient to replace symbolic approaches throughout students' mathematical experiences. As problem complexity increases (i.e., from simple two-step relations, to relations in which the unknown occurs twice, or from linear to nonlinear relations), an advantage of symbolic representations arises, surpassing the effectiveness of verbal representations (Koedinger, Alibali, & Nathan, in press; Nathan, Stephens, Masarik, Alibali, & Koedinger, 2002). Formal symbolic representations such as equations or graphs appear to "scale up" far better than verbally based solution methods as complexity increases. However, little is known about this developmental transition or the conditions that facilitate or hamper it.

Typically, symbolic reasoning is seen as the by-product of more general maturational processes (e.g., Santrock, 2001). Our take differs substantially from this hypothesis. We do not conceive the development of abstract mathematical reasoning as a natural process with a predetermined outcome. Instead, we argue that the ability to comprehend and use formal representations and methods is the result of carefully engineered learning experiences that connect to students' prior conceptions to invoke conceptual, procedural, and meta-cognitive knowledge in constructive and opportune ways.

TIER 2: MIDDLE SCHOOL ALGEBRA TEACHING PRACTICES

Research at the elementary grades suggests that instruction is more effective when teachers are familiar with students' preconceptions (Carpenter et al., 1996; Cai, 1998; Stacey & MacGregor, 1997). Teachers with a "cognitively based perspective" think that children construct their own mathematical knowledge—that skills are best taught within problem-solving contexts, and that instruction should be developmentally informed and organized to facilitate students' construction of knowledge (Peterson, Carpenter, & Fennema, 1989).

However, many teachers of mathematics reveal different views. Recent evidence suggests that both practicing and in-service teachers with advanced mathematics knowledge systematically misjudge the range and efficacy of students' informal solution strategies for algebra, and tend to assume students have proficiency with formally taught symbolic reasoning and solution methods (Nathan & Koedinger, 2000c; Nathan & Petrosino, 2003).

The study of teacher knowledge, beliefs, and practices is crucial to understanding and improving pedagogy and classroom experiences (Hashweh, 1996; Nespor, 1987). Research within this domain demonstrates that teachers' beliefs and knowledge are deeply rooted and often resistant to change (see, e.g., Brown, Cooney, & Jones, 1990; Manouchehri, 1997; Thompson, 1992).

We must also consider the potential impact of curricular innovation and reform on middle school algebra teaching and learning. A growing number of school districts are adopting reform-based curriculum programs such as those supported by NSF and NCTM (e.g., *Connected Mathematics* and *Mathematics in Context*). Although positive anecdotes abound, empirical studies have only begun to explore the dynamics of curricular reform, its impact on teaching practice and student achievement, and its interaction with teachers' knowledge and beliefs (Fuson, Carroll, & Drueck, 2000; Huntley, Rasmussen, Villarubi, Sangtong, & Fey, 2000). Borrowing from the work of Nancy Atwell (1991), Villaume (2000) noted the "*terrible freedom* that some teachers experience as they realize they are being asked to stop teaching programs and start teaching based on what children are thinking and doing" (p. 21). This is particularly true within the algebra domain, as transitions from conceptual to abstract understanding are regularly the desired outcome of learning.

TIER 3: RESEARCH ON TEACHER LEARNING AND TEACHER CHANGE

Middle school mathematics teachers often have little post-secondary mathematics training. As schools push for more complex mathematics in earlier grades, the disparities between the needs of students and the training of teachers become greater. We conceptualize teacher change as more than just adopting new practices. Ideally, change should engender continued professional growth that is self-sustaining and generates knowledge and reflection (Franke, Carpenter, Levi, & Fennema, 2001). Our goal for teacher professional development (TPD) is consistent with those of other researchers (e.g., Darling-Hammond & Ball, 1997; Lieberman, 1996; Loucks-Horsley, Hewson, Love, & Stiles, 1998; Putnam & Borko, 2000). One successful effort in mathematics is Cognitively-Guided Instruction (CGI) (Carpenter, Fennema, & Franke, 1996; Wilson & Berne, 1999). Excellent examples of such approaches are found within federally funded design experiments, in which teachers and researchers work closely together in classrooms (Bereiter, 2002).

While some research suggests that the mere introduction of new curricula can serve as a change lever, we hypothesize that meaningful teacher change results from a complex and sustained series of interactions that provide ample opportunities for experimentation, reflection, and intensive engagement in a professional community (Putnam & Borko, 2000). Hence, effective and lasting teacher change requires structured professional development programs to support implementation efforts. This calls for systematic and tight coordination between curriculum innovation, student reasoning, and TPD opportunities.

A drawback to such models, however, is that they are extremely resource-intensive and do not persist on a large scale when major funding is absent (Bereiter, 2002). In response to a pressing need for affordable and viable alternatives to large-scale implementation, many researchers, developers, educational institutions, funding agencies, and businesses are turning to a model of TPD that combines facilitated video case instruction (VCI) with online support for teachers. In VCI, teachers learn through group study and analysis of video cases that are often presented as stories of exemplary classroom lessons and examples of student learning and performance. This approach has been implemented in an online learning environment called eSTEP (Derry, 2006; Derry, Seymour, Steinkuehler, Lee, & Siegel, 2004; Derry, Siegel, Stampen, & the STEP Research Group, 2002) and TPD program of courses (Wortham, Wilsman, Derry, & Woods, 2004). Opportunities for teachers to view and discuss cases of classroom practice can improve instruction (e.g., Merseth, 1996; Shulman, 1992), serve as a vehicle for dissemination of reformed practice, and offer visions of what is possible (Shulman, 1992). Results from the Trends in International Mathematics and Science Study (TIMSS) Video Studies (Stigler & Hiebert, 1999), coupled with the release of videotapes of eighth-grade mathematics and science lessons from several countries, has helped popularize the use of video cases of classroom practice as a basis for TPD in mathematics and science.

In addition, there are compelling empirical and theoretical arguments for structuring TPD around video from teachers' classrooms. Using video from their own classrooms situates the exploration of teaching and learning in a more familiar, and potentially more motivating, environment than does using video from unknown teachers' classrooms (Seidel et al., 2005). As LeFevre (2004) pointed out, video makes teachers' classrooms accessible in a way that other media simply cannot, and therefore has the potential to be a powerful catalyst for change and improvement. Professional development leaders can select video excerpts to address particular features of teaching and learning that they want to examine, and the video can be stopped, replayed, or otherwise manipulated to focus conversations on those features.

SYNERGY AMONG TIERS

Our intention is to provide a coherent chain of reasoning from theory, to research questions, to data, to analysis, to evidence, and back to theory. As was initially conceptualized, elements of one thread would be developed to a sufficient degree that they could serve as inputs, constraints, and generative design guidelines to other inter-related tiers. Similarly, development challenges and successes could elicit research activities in connected tiers. Figure 2 illustrates the dynamics of how research and development from various tiers contributes information to and elicits information from other interdependent areas. Research on student performance assessment contributes to professional development design which then feeds into (and is informed by) our emerging models of the nature and relation of teacher knowledge and beliefs. Advances in understanding teacher knowledge provide insights in instructional practice and teacher change that influence student reasoning and learning, which can be documented in student assessments.

One of the strengths of our framework (Figure 1) is that all threads of research and development are highly interdependent. This interdependence is also one of the greatest challenges of conducting this kind of work. Two significant challenges of this approach that we encountered were time and communication. With regard to time, results often could not be produced and digested quickly enough by tier-specific teams to be sufficiently developed for a "down stream" need by another investigative team. Cross-tier communication also proved to be difficult, as

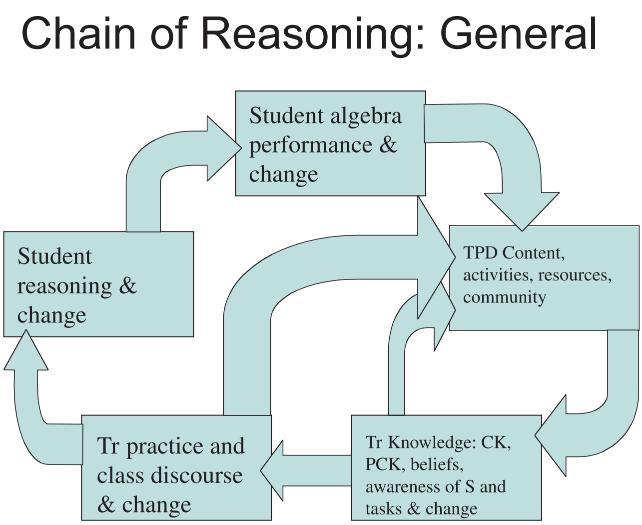

FIGURE 2 Diagram showing the dynamics of how research and development from various tiers contributes information to and elicits information from other interdependent areas. From the top, clockwise: Research on student assessment performance contributes to professional development design that then feeds into (and is informed by) our emerging models of the nature and relation of teacher knowledge and beliefs. Advances in understanding teacher knowledge provide insights into instructional practice and teacher change that influence student reasoning and learning, which can be documented in student assessments.

the various research teams strove to share ideas across research traditions, vocabulary, methodological approaches, and philosophical underpinnings. Even well-understood phenomena such as the verbal advantage reviewed above had to be recast from cognitive terms to sociocultural and practitioner terms so that they could be appropriately incorporated into future professional development program designs.

Our efforts to foster synergy among tiers included:

- Tier 1: Monthly cross-campus meetings on “What is Algebra?”, with a three-day meeting at the University of Wisconsin (Dec. 2001).
- Cross-tier annual meetings hosted by the University of Colorado.
- Tier 3: Two TPD meetings hosted by the University of Wisconsin (Dec. 2002) on video case-based reasoning and electronic learning environments for TPD, eSTEP, and Socio-Technical Environments for Learning and Learning Activity Research (STELLAR).
- Cross-tier/cross-campus monthly phone meetings.

While time and communication pose substantial obstacles, research efforts such as this can produce a large amount of new information about the nature of student thinking, instruction, and teacher change. The articles that follow each take up significant research questions within this multi-tiered framework and make strides to provide a systemic picture for implementing a program designed to understand and cultivate students’ transition from arithmetic to algebraic reasoning during the middle school years.

Contributions to This Issue

Working off of the grounding provided henceforth, this special issue provides results from each of the three tiers of the STAAR project. Tier 1 papers move forward with the previously mentioned hypothesis—that the ability to comprehend and use formal algebraic representations and methods is the result of carefully engineered learning experiences that connect with students’ prior conceptions to invoke conceptual, procedural, and meta-cognitive knowledge in opportune ways. The two articles presented here further examine the manner in which students gain conceptual and procedural fluency in algebraic reasoning and add to the body of literature of developmental landmarks that prove important in students’ understanding of algebra.

The contribution by Alibali, Knuth, Hattikudur, McNeil, & Stephens examines students’ understanding of the equal sign, and specifically, the ways in which students solve *equivalent equations problems* and then develop more sophisticated reasoning over time. Results indicate that a more sophisticated understanding of the equal sign is associated with better performance on equivalent equations problems. Although student trajectories of equal sign understanding varied, performance on equivalent equation problems was related to when students' acquired a more sophisticated level of understanding of the equal sign. Interestingly, those who acquired a relational understanding earlier were more successful at solving the equivalent equation problems at the end of grade 8. These findings provide further insight into the learning experiences that students need in terms of gaining a solid understanding of the equal sign. In particular, this article underscores the fact that elementary and middle school teachers need to support a foundational understanding of equivalence from a young age.

Equivalence is just one facet of algebraic knowledge that is central to middle school students. Nathan and Kim's article examines representational fluency and the strategies students implement when they demonstrate conceptual, procedural, and meta-cognitive understandings of pattern generalization. The authors analyze middle school students' performance and strategy use to determine how students solve pattern generalization problems and which strategies contributed to successful solutions. Most notable was their finding that students show greater success with patterns presented in a continuous format, such as line graphs and verbal rules, compared to problems presented as a collection of discrete instances, such as a piece-wise graph. In addition, Nathan and Kim's research indicates that students perform better on pattern generalization tasks when words and graphs are combined. These findings contribute to our understanding of the ways in which students are able to move among and between representations and, combined with previous research, provide an emerging developmental model of representational fluency.

The Tier 2 article included in this issue focuses on teachers' knowledge of students' understanding of algebraic concepts, specifically students' understanding and use of the equal sign and variable (Asquith, Stephens, Knuth, & Alibali). This article is directly aligned with Tier 1 results and illustrates the unique connections within this project using the multi-tier framework. Specifically, in this study, middle school teachers predicted student responses to written assessment items focusing on the equal sign and variable. Teachers' predictions of students' understanding of the variable were aligned to a large extent with students' actual responses to corresponding items. On the other hand, teachers' predictions of students' understanding of the equal sign did not correspond well with actual student responses. These findings indicate a need for professional development with teachers to support students' conceptual development of the equal sign and, to a lesser extent, variable.

The Tier 3 articles draw on the work of the previous tiers in developing novel professional development programs. One of the major long-term aims of our research is to assist a large number of teachers to acquire a deeper understanding of algebra, algebraic reasoning, and new practices in the teaching of algebra. Thus, an important component of our work was to design, implement, and evaluate prototype "proof-of-concept" programs of teacher professional development within a sample of Colorado and Wisconsin schools. Two professional development programs were implemented at two different sites: Madison, Wisconsin and Boulder, Colorado. These programs were designed to help in-service teachers support student development of algebraic reasoning.

The first Tier 3 article describes a professional development model, the Problem-Solving Cycle (PSC), that evolved with a group of teachers over a two-year period to support the development of content and pedagogical content knowledge in the area of algebra (Koellner, Jacobs, Borko, Schneider, Pittman, Eiteljorg, Bunning, & Frykholm). The PSC model consists of three interrelated professional development workshops. The first workshop concentrates on fostering teachers' content knowledge related to a specific task that they will then implement in their classrooms. The second and third workshops focus on the role of the teacher and students' algebraic thinking and rely heavily on video clips from the teachers' lessons.

The next article describes the use of contrasting cases to teach a university course focused on helping teachers to support their students' transition from arithmetic to algebraic reasoning (Derry, Wilsman, & Hackbarth). The researchers designed contrasting-case instructional activities that encouraged teachers to interpret and compare multiple representations and solutions of mathematical tasks in both their own work and the work of their students. Both of these professional development projects employed a unique structure to support teacher development in the area of content knowledge and pedagogical content knowledge. In addition, both projects built on research findings from Tiers 1 and 2 including, but not limited to, a more nuanced understanding of students' and teachers' knowledge about (a) the equal sign, (b) representational fluency, and (c) patterns and functions.

REFERENCES

Atwell, N. (1991). Side by side: Essays on teaching to learn. Portsmouth, NH: Heinemann.

Bereiter, C. (2002). *Education and the mind in the knowledge age.* Mahwah, NJ: Erlbaum.

Bransford, J. D., Brown, A. L., & Cocking, R. (1999). *How people learn*. Washington, DC: National Academy Press.

Brown, C., Cooney, T., & Jones, D. (1990). Mathematics teacher education. In W.R. Houston (Ed.), *Handbook of research on teacher education* (pp. 639–656). New York: Macmillan.

Cai, J. (1998). Developing algebraic reasoning in the elementary grades. *Teaching Children Mathematics, 5*, 225–228.

Carpenter, T. P., Fennema, E., & Franke, M. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal, 97*, 3–30.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Heinemann.

Case, R., & Okamoto, Y. (1996). *The role of central conceptual structures in the development of children's thought* (Monographs of the Society for Research in Child Development, No 24). Author.

Darling-Hammond, L., & Ball, D. L. (1997, June). Teaching for high standards: What policymakers need to know and be able to do. National Educational Goals Panel. Available at http://www.negp.gov/Reports/highstds.htm

Derry, S. J. (2006). eSTEP as a case of theory-based Web course design. In A. M. O'Donnell, C. E. Hmelo-Silver, & G. Erkens (Eds.), *Collaborative reasoning, learning, and technology* (pp. 171–196). Mahwah, NJ: Erlbaum.

Derry, S. J., Seymour, J., Steinkuehler, C., Lee, J., & Siegel, M. (2004). From ambitious vision to partially satisfying reality: An evolving sociotechnical design supporting community and collaborative learning in teacher education. In S. A. Barab, R. Kling, & J. H. Gray (Eds.), *Designing for Virtual Communities in the Service of Learning* (pp. 256–295). Cambridge: Cambridge University Press.

Derry, S. J., Siegel, M., Stampen. J., & the STEP Research Group (2002). The STEP system for collaborative case-based teacher education: Design, evaluation and future directions. *Proceedings of Computer Support for Collaborative Learning 2002, Boulder, CO.*

Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teachers' generative growth: A follow-up study of professional development in mathematics. *American Educational Research Journal, 38*, 653–689.

Fuson, K. C., Carroll, W. M., & Drueck J. V. (2000). Achievement results for second and third graders using the standards-based curriculum everyday mathematics. *Journal of Research in Mathematics Education, 31*, 277–295.

Hashweh, M. Z. (1996). Effects of science teachers' epistemological beliefs in teaching. *Journal of Research in Science Teaching, 33,* 47–63.

Huntley, M., Rasmussen, C., Villarubi, R., Sangtong, J., & Fey, J. (2000). Effects of standards-based mathematics education: A study of the Core-Plus Mathematics Project Algebra and Functions Strand. *Journal for Research in Mathematics Education*, *31*, 328–361.

Kalchman, M., & Case, R. (1998). Teaching mathematical functions in primary and middle school: An approach based on neo-Piagetian theory. *Scientia Paedagogica Experimentalis*, *35*, 7–54.

Kaput, J. (May, 1998). *Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K–12 curriculum.* Paper presented at the Algebra Symposium, Washington, DC.

Kaput, J., Carraher, D., & Blanton, M. (2007), *Algebra in the Early Grades.* London: Taylor & Francis.

Kieran, C. (1988). Two different approaches among algebra learners. In A. F. Coxford (Ed.), *The ideas of algebra, K–12* (1988 Yearbook, pp. 91–96). Reston, VA: National Council of Teachers of Mathematics.

Knuth, E. (2000). Student understanding of the Cartesian Connection: An exploratory study. *Journal for Research in Mathematics Education*, *31*, 500–508.

Koedinger, K. R., Alibali, M., & Nathan, M. J. (in press). Trade-offs between grounded and abstract representations: Evidence from algebra problem solving. *Cognitive Science*.

Koedinger, K. R., & Nathan, M. J. (2004). The real story behind story problems: Effects of representations on quantitative reasoning. *Journal of the Learning Sciences, 13*, 129–164.

Ladson-Billings, G. (1998). It doesn't add up: African American students' mathematics achievement. In Carol E. Malloy & Laura Brader-Araje (Eds.), *Challenges in the mathematics education of African American children: Proceedings of the Benjamin Banneker Association Leadership Conference* (pp. 7–14). Reston, VA: National Council of Teachers of Mathematics.

LeFevre, D. M. (2004). Designing for teacher learning: Video-based curriculum design. In J. Brophy (Ed.), *Advances in research on teaching: Vol. 10. Using video in teacher education* (pp. 235–258). Oxford, UK: Elsevier.

Lesh, R., & Kelly, A. (2000). Multi-tiered teaching experiments. In Kelly, A. & Lesh, R. (Eds.). *Research Design in Mathematics & Science Education* (pp. 197–230). Kleuwer: Amsterdam, Netherlands.

Lieberman, A. (1996). Creating intentional learning communities. *Educational Leadership, 54*, 51–55.

Loucks-Horsley, S., Hewson, P.W., Love, N., & Stiles, K.E. (1998). *Designing professional development for teachers of science and mathematics.* Thousand Oaks, CA: Corwin Press Inc.

Manouchehri, A. (1997). School mathematics reform: Implications for mathematics teacher preparation. *Journal of Teacher Education, 48,* 197–209.

Merseth, K. K. (1996). Cases and case methods in teacher education. In J. Sikula (Ed.), *Handbook of research on teacher education* (pp. 722–744). New York: Macmillan.

Nathan, M. J., & Koedinger, K. R. (2000a). Moving beyond teachers' intuitive beliefs about algebra learning. *Mathematics Teacher, 93*, 218–223.

Nathan, M.J., & Koedinger, K. R. (2000b). Teachers' and researchers' beliefs about the development of algebraic reasoning. *Journal for Research in Mathematics Education, 31* , 168–190.

Nathan, M. J., & Koedinger, K. R. (2000c). An investigation of teachers' beliefs of students' algebra development. *Cognition and Instruction*, *18*, 209–237.

Nathan, M. J., & Petrosino, A. J. (2003). Expert blind spot among preservice teachers. *American Educational Research Journal*. *40*, 905–928.

Nathan, M. J., Stephens, A. C., Masarik, D. K., Alibali, M. W., & Koedinger, K. R. (2002). Representational fluency in middle school: A classroom study. In D. S. Mewborn, P. Sztajn, D. Y. White, H. G. Wiegel, R. L. Bryant, & K. Nooney. (Eds.) *Proceedings of the Twenty-Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Vol. 1* (pp. 462–472). Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Nespor, J. (1987). The role of beliefs in the practice of teaching. *Journal of Curriculum Studies, 19,* 317–328.

National Council of Teachers of Mathematics (NCTM). (2000). *Curriculum and evaluation standards for school mathematics* (p. 5). Reston, VA: NCTM.

National Research Council (NRC). (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

NRC. (2005). *The nature and role of algebra in the K–14 curriculum.* Washington, DC: National Academy Press.

Peterson, P. L., Carpenter, T. P., & Fennema, E. (1989). Teachers' knowledge of students' knowledge in mathematics problem solving: Correlation and case analysis. *Journal of Educational Psychology, 81*, 558–569.

Putnam, R., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teaching? *Educational Researcher, 29*, 4–15.

RAND (2003). *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education.* RAND Mathematics Study Panel (D. Ball, Chair).

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology, 91*, 175–189.

Stacey, K., & MacGregor, M. (1997). Ideas about symbolism that students bring to algebra. *Mathematics Teacher, 90*, 110–113.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: Simon and Schuster.

Shulman, J. (Ed.) (1992). *Case methods in teacher education*. New York: Teachers College Press

Swafford, J., & Langrall, C. (2000). Grade 6 students' preinstructional use of equations to describe and represent problem situations. *Journal for Research in Mathematics Education, 31*, 89–112.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127–146). New York: Macmillan.

Villaume, S. K. (2000). The necessity of uncertainty: A case study of language arts reform. *Journal of Teacher Education, 51*, 18–25.

Wilson, S. M., & Berne, J. (1999). Teacher learning and the acquisition of professional knowledge: An examination of research on contemporary professional development. In A. Iran-Nejad & P. D. Pearson (Eds.). *Review of research in education* (Vol. 24, pp. 173–209). Washington, DC: American Educational Research Association.

Wortham, D. L., Wilsman, M., Derry, S., & Woods, D. (2004, April). *Design of professional development: STELLAR lab's pragmatist perspective.* Paper presented at the annual meeting of the American Educational Research Association, San Diego, CA.