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To Disagree, We Must Also Agree: How Intersubjectivity Structures and Perpetuates Discourse in a Mathematics Classroom

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Learning in a socially mediated context like a classroom places emphasis on the ability of learners to communicate their ideas to others, and for members of a class to achieve shared meaning or intersubjectivity (IS). We take a participatory view of IS, where both consensual agreement and disagreement are regarded as aspects of a common set of processes that mediate collective activity. Interlocutors need not demonstrate convergence toward a common idea or solution to exhibit IS and, indeed, they appear to need a shared understanding to express substantive disagreement through divergent views. Multilevel, multimodal analyses of videotape of a middle school mathematics classroom, including speech, gestures, drawing, and object use, reveal a discourse that is organized into recurrent sequences of event triads. The dynamics toward and away from convergent ideas appears to be instrumental in fostering sustained and engaging discourse and influencing the representations that students propose during problem solving. Participants frequently exhibited IS, but, as allowed for in the participatory view, the interactions did not seem to convert many students from their initial interpretations. Instead, disagreements and a desire to establish common understanding appeared to lead participants to express their diver-

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gent views in more refined and accessible ways. Advancement of our understanding of the role that IS serves in socially mediated learning has the potential to inform both educational theory and emerging areas in embodied cognition and cognitive neuroscience that addresses imitation and empathy, and thus help to bridge research between brain function and social cognition.

Theories of learning in social settings, no matter their philosophical orientation, must address issues of intersubjectivity (IS). Sociologists such as Schegloff (1992) place IS foremost in addressing any and all aspects of social interaction: “The problem of intersubjectivity (or cognitive order) is theoretically anterior to whatever formulations of problems of order or conflict are part of the tradition of social theory” (p. 1296). Without IS, Schegloff argued, the entire enterprise of social science stands without any reference to the world it purports to identify or describe. Psychologists such as Clark (1996) regard all forms of communication as a way to ground meaning in both the cognitive and social realms. IS also plays a significant role for sociocultural theorists such as Vygotsky (1986) and Lerman (1996, 2000), who consider it to be at the heart of learning and of consciousness itself. Most recently, Suthers (2006) argued for an expanded focus on studies of intersubjective meaning making in laying out an agenda for the future of research on computer-supported collaboration. With such notable scholars affording such a prominent place to IS, it is important to clarify what IS refers to, how it is manifest, and how it functions as an influential force for the structure and dynamics of classroom discourse.

The archetypal account of IS appears to be the story *The Blind Men and the Elephant*. Although the origin of the story is in some dispute (it is reported as an ancient tale stemming from both Buddhist and Jain cultures) and there are many versions to be found, the essential elements of the story are these: A group of men, blind since birth, encounter an elephant, though each experiences only a part of the animal. Each man asserts that the entire elephant is as its part: The elephant is a rope (tail), a spear or ploughshare (tusk), a tree (leg), and so on. In his rendition, the American poet John Godfrey Saxe (1816–1887; Brooks, 1906) wrote as follows:

> Each in his own opinion
> Exceeding stiff and strong,
> Though each was partly in the right,
> And all were in the wrong!

The story is, of course, an allegory of IS. Consensus is thwarted because of the men’s limited and differing perspectives—the blindness that we all have. All that is needed, seemingly, is a shared understanding of the elephant, made possible if the blind men could just “see” the elephant as it really is.

This article starts with the premise that IS is a fundamental and unavoidable aspect of social interaction, and that understanding its nature is necessary for developing reliable theories of socially mediated learning and for designing the next generation of effective learning environments. In this article, we show that IS can be regarded as broader than agreement or consensus (Matusov, 1996, 2001) and can provide insights into participants' interactions more generally, including their disagreements, divergence of ideas and solutions, and misunderstandings in the constructivist classroom. Our central hypothesis is that IS acts as more than a point of convergence toward a common idea or solution, but that the dynamics toward and away from convergent ideas appear to be instrumental in fostering sustained and engaging discourse and influencing the representations that students propose during problem solving. We use discourse analytic techniques to show how IS is manifest in the classroom and to explore its role in structuring and perpetuating participants' intellectual interactions.

INTERSUBJECTIVITY

Traditional views have tended to equate IS with consensual agreement and present IS as an attribute of a group activity or discussion that a group either succeeds in achieving or fails to achieve. Success, in this view, means that participants have acquired a shared understanding (Cole, 1991; Stahl, 2006) or univocality (Lotman, 1988). Efforts by interlocutors, such as conversational repair, constitute a normal and critical aspect of dialogue as participants strive to address obstacles to their mutual understanding (Schegloff, 1992). In the traditional view, IS is reduced to a single subjectivity among participants (Matusov, 1996). Some researchers have examined measures of convergence among interlocutors as the overall movement toward or away from a common goal (Kapur, Voiklis, Kinzer, & Black, 2006). Typically, evidence for IS within traditional and more contemporary views has shown movement from a state of disagreement or misunderstanding to one of agreement or symmetry (e.g., Wertsch, 1979).

There have been several challenges to the view that consensual agreement and convergence toward a common idea capture the essence of IS. First, some scholars have expressed concern about the strong value judgment that deems agreement as favorable and disagreement as unfavorable (Smolka, de Goes, & Pino, 1995). This bias is problematic, because the important role of disagreement in cognitive development and socially mediated learning is well established (e.g., Johnson & Johnson, 1989; Piaget, 1975/1985; Posner, Strike, Hewson, & Gertzog, 1982; Vygotsky, 1978). Second, some have contested the traditional view of IS that casts disagreement and agreement as separate states or phases along a developmental progression (e.g., Wertsch, 1979) and dismisses their complementary nature (Smolka et al., 1995) and frequent coexistence (e.g., Matusov, 1996). Third, it can be argued that the traditional view is too narrow in suggesting that the processes that are unique to IS are no longer in play in failed IS. This traditional perspective further distances those processes that mediate disagreement and disequilibrium from those that mediate agreement.

Alternative accounts of IS have fueled some reevaluation. For example, Steffe and Thompson (2000), articulating the radical constructivist view, took a nuanced approach to the process and outcome of IS. On the one hand, their perspective was consistent with the idea that interlocutors reach some form of convergence as part of the process of establishing IS. On the other hand, they distinguished their perspective from the traditional one by emphasizing the reciprocal interactions that are achieved: "By *reaching mutual agreement* we do not mean that the interacting individuals end up with the identical conceptual structures. Rather, we mean only that their conceptual structures are sufficiently compatible for successful reciprocal assimilation" (p. 193, italics in the original).

In practice, of course, speakers do not attain identical conceptual structures. In some of the seminal thinking in this area, Rommetveit (1985) posited that communication affords "states of partial intersubjectivity" that allow speakers to temporarily bridge their "private worlds" (p. 185). This is consonant with models of IS emerging from cognitive neuroscience that address basic social processes such as imitation and empathy. Researchers such as Gallese (2003a; Rizzolati, Fogassi, & Gallese, 2001) have argued that there are specific neural mechanisms, called *mirror neurons*, through which individuals directly understand the actions of others because these actions evoke in people the bodily states that they would normally occupy if they had initiated those same actions. From this empathetic response, speakers constitute *a shared manifold of intersubjectivity* with those with whom they interact (Gallese, 2003b). This allows interlocutors to provisionally enter into a shared social space, even while their ideas and interpretations may differ from one another.

The participatory view of IS (Matusov, 1996) attempts to integrate the view of mutually shared space with the traditional view. The participatory view of IS focuses on "the coordination of individual participation in joint sociocultural activity rather than as a relationship of correspondence of individuals' actions to each other" (p. 26). Within this view, agreement and disagreement are considered aspects of a common set of processes that mediate collective activity. Interlocutors need not reach consensus to exhibit IS. They can converge on some aspects and diverge on others (Matusov & White, 1996). For example, a speaker may appropriate the representation of a peer but regard it through an alternative interpretive frame. In this way, the participatory view distinguishes between establishing a shared space of interaction and establishing consensus.

One of the greatest challenges for IS is to span different frames of reference (Bateson, 1972), or speech genres (Bakhtin, 1986), in order to foster effective communication. Mortimer and Wertsch (2003) explored how dialogue in a Brazilian science classroom involved the negotiation of IS for topics that invoked both empirical (everyday) and theoretical (scientific) speech genres. For example, in a lesson on the particulate nature of matter, students and their teacher negotiated the meaning of *gas*. For typical students, the term invoked their everyday experiences with bottled liquid fuel (butane) used for cooking. However, the teacher, employing a theoretical speech genre, was referring to a hypothetical class of entities with no fixed shape or volume. Mortimer and Wertsch described the classroom conflict as rooted in these misaligned genres:

> These lessons can be said to represent a different contract of intersubjectivity, set up with the implicit admission that multiple voices, which could even be in conflict, should be taken into account as part of this only partially shared world populated by different ways of categorizing matter. (pp. 237–238)

As Mortimer and Wertsch (2003) suggested, a refined theory of IS ("a different contract") should be able to address such disparate notions within the discourse. In fact, rich dialogue may actually thrive when alternative interpretations and disagreement operate within a shared context. For example, in their extended study of classroom discourse in inquiry-based science classrooms, Wells and Arauz (2006) found that sustained dialogic interactions tended to arise out of the differences of opinion.

SOCIALLY MEDIATED LEARNING IN THE CLASSROOM

Constructivist approaches to classroom instruction draw heavily on students' own conceptions. For this reason, IS is evident in several studies of socially mediated learning and practice that operate within the constructivist paradigm (e.g., Cobb, Yackel, & Wood, 1993). Lerman (1996) argued that IS is constituted through social practices and socially mediated activity. In Vygotsky's (1978) theory of social development, speech, writing, and other social tools serve to mediate social interaction. Such tool usage also serves as a mediator of participants' cognitive development (Wertsch & Sohmer, 1995), as people internalize the tools' physical and cognitive functions, which then contributes to the construction of higher mental processes.

Recent education reform has adopted some of the principles of socially mediated learning as a means to promote higher order thinking in all subject areas, including reading (Palinscar & Brown, 1984), science (Brown & Campione, 1994; Palincsar & Magnusson, 2000; Songer, 2004; van Zee & Minstrell, 1997), teacher education and professional development (Grossman, Wineburg, & Woolworth, 2001; Matusov, 2001; Palincsar, Magnusson, Marano, Ford, & Brown, 1998), and mathematics (Ball, 1996; Cobb et al., 1993; Cognition and Technology Group at

Vanderbilt, 1997; Lehrer, Strom, & Confrey, 2002). Current mathematics education standards, for example, call for an emphasis on communication as one of the five process standards considered essential to acquiring and using mathematical knowledge (National Council of Teachers of Mathematics, 2000). Teachers struggle with their role as facilitators within this new learning environment (e.g., Nathan & Knuth, 2003; Rittenhouse, 1998). However, as teachers come to develop facility in their new role and learn how to manage classrooms that draw heavily on peer interactions and student-led presentations, they do see benefits (Cobb et al., 1993; French & Nathan, 2006).

It was within a setting of the early adoption of principles and practices of socially mediated classroom learning that we came to observe sixth graders and their teacher engaged in a spirited dialogue about a spatial reasoning task posed by one of the students. We call this task the Pie Problem: How do you cut a pie into eight equal-sized pieces making only three cuts? For most of the double period we observed, students worked out solutions, discussed them with peers, and then publicly presented their ideas, offered alternatives, and critiqued and elaborated their proposed solutions to the Pie Problem.

We focused considerable attention on the representations produced by the class participants because it was through these that students conveyed their analytical ideas about the problem and about their reasoning and interpretations of the problem context. From a pedagogical perspective, the public display of solution representations supported tenets of social constructivism that acknowledge the collaborative cocreation of mathematics. It also provided occasions for "teaching moments" to address mathematical ideas that may have been presented either correctly or incorrectly in the course of the group interaction.

The examination of representations also was a natural way to consider whether convergence toward a common solution representation, as would be expected from a traditional view of IS, was the proper way to describe the discourse. However, rather than witnessing convergence, we observed students refining their ideas and uses of representations to suit their interpretive frames. In the end, there was no clear convergence. Yet we argue that there was a great deal of IS among participants, and this was a major force shaping the extended discourse.

ANALYSIS OF CLASSROOM DISCOURSE

Institutional talk such as classroom conversation (Drew & Heritage, 1992) shares many characteristics with ordinary conversation, but it also exhibits some unique properties. Both qualitative and quantitative discourse analysis methods can be brought to bear and applied at multiple levels to understand the nature of the classroom discussion and to identify the elements of the ensuing interactions and the dynamics that drove them.

Methods of Coding and Unitization

One of the most common coding schemes for depicting sequences of classroom discourse highlights the *initiation–response–evaluation* (IRE) patterns (Mehan, 1979; Sinclair & Coulthard, 1975). In a typical IRE pattern, the teacher asks a closed question or invites student input, which elicits a reaction from a student, whose response is then evaluated by the teacher, often in a way that terminates the interaction ("That is incorrect" or "Correct!"). In one common modification, IRF, the evaluation phase is replaced with a follow-up question that tends to perpetuate the IRF/IRE pattern (Wells, 1993; Wells & Arauz, 2006).

Each of these events (initiation, response, etc.), and others like them, are coded from the discourse transcript and analyzed or interpreted through some theoretical perspective. Code-based methods of qualitative discourse analysis allow for the exploration of relationships between emergent or theory-driven themes that are depicted by researcher-constructed concepts and categories. In addition, *quantitative content analysis* (Carley, 1990; Chi, 1997; Krippendorf, 2004) focuses on the statistical analysis of patterns of concepts and categories. Combining these approaches allows one to pursue hypothesis testing as well as descriptive approaches of inquiry.

Both qualitative and quantitative approaches of discourse analysis commonly rely on unitization schemes to segment the data into analytically comparable elements. Meaning and usage are useful determiners of the unitization of extended discourse. Gee (2005) suggested using *stanzas* to describe the units that compose a discourse. Stanzas in a transcript can be regarded as similar to paragraphs in an essay. Gee (2005) described:

> Each stanza is a group of lines about one important event, happening, or state of affairs at one time and place, or it focuses on a specific character, theme, image, topic, or perspective. When the time, place, character, event, or perspective changes, we get a new stanza. (p. 109)

Multilevel Perspectives of Discourse

Studies of classroom discourse can convey the complex and adaptive nature of the interactions that shape group learning and collaborative problem solving (e.g., Lampert & Blunk, 1998; Peressini & Knuth, 1998; Schoenfeld, 1998). Classroom discourse can be studied at several interdependent but partially decomposable levels (Matusov, 1996; Nathan, Knuth, & Elliott, 1998; Wells & Arauz, 2006). For example, Nathan and Knuth (2003) looked at how considerations such as teacher beliefs and goals, current education reform demands, and opportunities to reflect on one's emerging teaching practices influenced classroom instruction over a 3-year period. The details of this complex relation were not apparent, however, without also examining mutually constraining levels of analysis that addressed (a) the moment-to-moment (micro level) flow of information among the members of the discourse community, (b) the nature and purposes of classroom scaffolding (at the meso level), and (c) global patterns of interaction that occurred across an entire discourse (macro level). Analyses along mutual but partially decomposable levels allow one to focus on certain phenomena while still providing a relatively integrated account of the behavior captured by the data.

Research Focus

Although the existence and importance of IS is well documented, the way in which IS transpires in discourse-based classrooms and the role it plays in shaping social interactions are less well understood. We used both quantitative content analysis and qualitative analysis methods in our investigation. In addition, we drew on a multilevel framework for organizing our current views on the data and establishing our research questions. However, we tried to be responsive to the particularities of this data set in determining the levels of analysis that structured our inquiry. The *global level* considered changes over the entire discourse. One of our central foci was the changing use of solution representations. We asked: How does the discourse unfold over the course of the class? How does representation use change over the discourse? What role does IS appear to play in these changes?

At the *meso level*, sitting between descriptions of the global progression of the discourse and the micro level view of individual turns and actions, we were interested in identifying and describing the discourse events among participants that revealed the nature of their dynamics. How is the discourse structured? What perpetuates the discourse? What role does IS play in influencing students' interpersonal interactions, including their uses of public solution representations and the subsequent reactions of the other participants?

In future work, we plan to report on analyses at the *micro level*, where the focus is on individual students' actions and utterances as they occur during each turn of the discussion. The focus in this article is on the global and meso levels, with the aim of analyzing the structure and dynamics of the discourse where IS appears to shape the classroom discourse most appreciably.

METHOD

Participants and Setting

Sixth-grade students in a middle-class community in the western United States engaged in solving the Pie Problem. One of the students, Manisha,[1] posed the prob-

[1]All participants' names have been replaced by pseudonyms to ensure confidentiality.

lem as follows: "How do you cut a pie into eight equal-sized pieces making only three cuts?" Manisha presented the problem to the teacher during the customary class warm-up activity. Finding the problem appropriate, the teacher invited Manisha to present it to the entire class. Students spent more than an hour out of a 90-min double period of their mathematics class solving this problem and discussing its solutions, first working individually, then in pairs, and then with the class as a whole. Our focus here is on the whole-class discussion.

The class normally had 24 students. We observed 20 students participating in the discussion, with 13 playing a particularly active role. The other 7 students involved were observed verbalizing their views clearly but indirectly, as part of a chorus of students.

The mathematical performance of the students in the class varied widely, with performance on the California Achievement Test ranging from the 5th to the 99th percentile. Five students in the class received special education support for physical and cognitive disabilities. A paraprofessional came once a week to help the teacher meet these students' special needs, though the aide was not present during the lesson under investigation.

The class session took place in late October, when school had been in session about 2 months. By this time, students were familiar with classroom norms for group participation and had spent considerable time publicly presenting their own mathematical ideas, posing questions to their peers, and practicing active listening skills in both mathematical and nonmathematical contexts.

Analytic Approach

Transcription and unitization. The classroom discourse was captured on video and digitized, then imported into Transana, a computer application for discourse analysis (Fassnacht & Woods, 2005). The video was first transcribed generally for utterances. At this time, we also used Transana to create an audio waveform file that visually illustrated the amplitude of sound over the time course of the video (Figure 1). The waveform is particularly useful for identifying pauses and especially active parts of the discourse. This initial "rough" transcript served as the record for viewing the videotape and fostering early hypothesis generation. The video and transcript were then analyzed over multiple passes in the manner similar to that suggested by Duncan (n.d.).

Multiple passes through the data. During the first pass, we unitized the transcript at the stanza level, identifying principal interactions between participants. As we conceptualized the criteria for determining stanza boundaries following Gee (2005), we started each stanza with the initiation of a new speaker's turn to present ideas about the problem and continued it until a substantively new idea or line of discussion was introduced. Stanzas in the transcript were bracketed by video time codes that allowed us to coordinate movement through one medium (e.g., video) with movement through the other (e.g., transcript). (Transana visually highlights the corresponding region of the transcript during video playback.) This allowed us to easily track the speech with the videotaped actions, and vice versa.

The second pass extended the initial transcript by including information on gestures and representation use. We also corrected any initial transcription errors and included any utterances that had seemed unintelligible during the first pass. The third pass focused on the particular representations used and coded them along three dimensions: (a) use of the principles of perspective, (b) effort to disambiguate the representation, and (c) internal consistency.

The fourth pass took a discourse analysis perspective, examining speech events within stanzas that were appropriate for our research questions at the meso level. Here we were guided by prior work on classroom discourse that identified common triadic sequences among interlocutors, such as the previously mentioned IRE patterns (Lemke, 1990; Mehan, 1979; Sinclair & Coulthard, 1975). Whereas stanzas may be regarded as paragraphs, addressing one complete interaction, events in our analyses may be likened to sentences.

The fifth pass identified utterances that conveyed both convergent (IS^+) and divergent (IS^-) IS (Matusov, 1996). We coded IS^+ whenever there was evidence that speakers shared a common frame of reference, such as speaking about a common

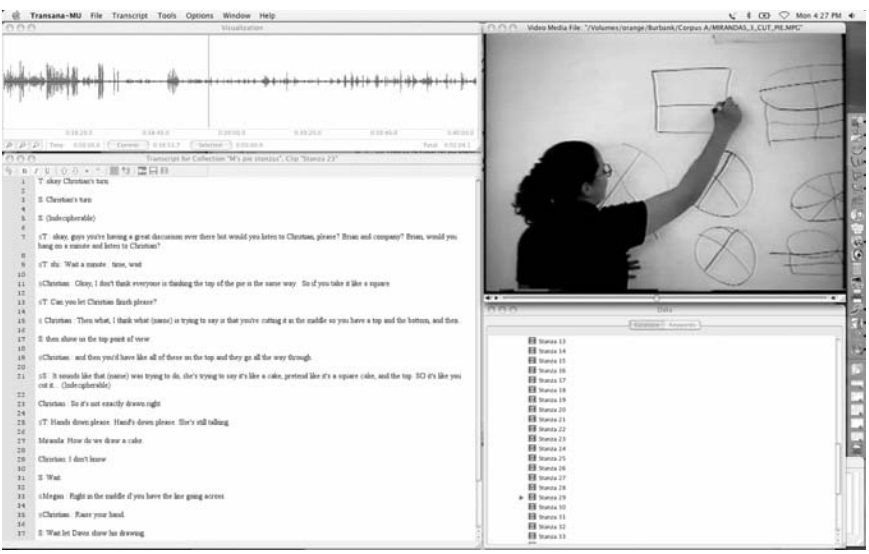


FIGURE 1 Screen image of the Transana software package depicting (clockwise from top left) the audio wave form, video, hierarchical database, and transcript windows.

representation or stating agreement. IS⁻ was coded when speakers showed disagreement, alternative interpretations, or confusion.

Reliability. We established interrater reliability for our unitization and coding practices. An individual who was familiar with video coding and the Transana software package, but who was unaffiliated with the research team and unfamiliar with our specific research questions, served as a reliability check. Codes were accompanied by video exemplars for training purposes. The external rater assigned codes and drew unit boundaries for approximately 10% of the video clips. We report Cohen's kappa measure of interrater reliability and percent agreement within each of the Results subsections.

RESULTS

The unitization process revealed 36 stanzas during the whole-class discussion of the Pie Problem, which ran for about 37 min and 30 s (with a classroom break midway through). The interrater reliability for stanza divisions based on the criteria stated above was found to be 100% agreement and no disagreement. As mentioned, stanzas were themselves composed of more basic events of participant interaction that distinguished between the initiation of a new topic, participant responses, and participant evaluations of those responses. Prior research on classroom communication showed the prevalence of IRE sequences and their variants (Greenleaf & Freedman, 1993; Lemke, 1990; Mehan, 1979; Sinclair & Coulthard, 1975). Several common events were observed within stanzas during our fourth pass through the transcript:

- *Initiation-closed event:* A known-answer question (Matusov, Bell, & Rogoff, 2002)—asked to involve or assess other speakers—that has a fixed or closed set of responses, often with a "best" response known by the speaker.
- *Initiation-open event:* An information-seeking question (Matusov et al., 2002) with no expected or "best" response known by the speaker.
- *Response:* Reply is a short, verbal response, typically following, though not dependent on, a known-answer question, as in an IRE sequence (Mehan, 1979).
- *Demonstration:* Reply contains a representation that offers a proposed solution in the form of a drawing, string of gestures, or manipulation of objects; and that typically follows, though is not dependent on, an initiation-open event.
- *Evaluation:* A value judgment in reference to a response or demonstration.
- *Elaboration:* An addition, modification, or query about a preceding demonstration or response.

Table 1 shows the frequencies of occurrence of these events. As can be expected in a socially mediated problem-solving setting such as this, demonstration, elaboration, and evaluation phases were common throughout the discourse and took up a majority of the time. In a surprisingly large number of cases (n = 24), the elaboration and evaluation phases were interlaced, often by the same speaker, leading us to apply both codes to the same events 85.7% of the time. For this reason, we opted to combine elaboration and evaluation events into a single E event category.

TABLE 1
Frequency of Events Coded From the Whole-Classroom Discourse

Stanza	*Code*	*n*	*Example*
Total stanzas		36	
Events	Initiation	30	
	Closed	2	T: Who would respond to Janet about nobody said the pieces are equal?
	Open	28	S2: Ben, draw yours on the board then.
	Response	2	S: Well, it says on the board how you 'cut a pie into eight equal-sized pieces.'
	Demonstration	28	S: Like I mean … who would want to have a pie that doesn't have like a bottom dress thing [*making a small circle with his right hand and then putting his left hand at the bottom with his right hand hitting the left hand several times*]
	Elaboration/evaluation	30	
	Elaboration[a]	24	S: If you eventually cut all the pieces, those pieces will fall into the cuts and then it'll be like a cut.
	Evaluation[a]	24	S: Well, cutting on the top of the pie that wouldn't be right. Well, ours is right.
IDE stanzas		28	
IRE stanzas		2	

Note. T = teacher; S = student.

[a]Of the 24 evaluation events and 24 elaboration events, 24 co-occurred. Elaboration occurred uniquely only four times, and evaluation occurred uniquely twice.

I Events: Who Directs the Discourse

An essential part of understanding the basic structure of a discourse is knowing who directs it. I events indicate when a speaker begins a new thread or invites others to contribute to the discourse. As noted above, initiation can be open or closed. Closed I events are common in certain instructional settings, but they made up only a small portion (5%) of the I events observed in this corpus. The prevalence of open I events was indicative of the dialogic nature of this class. Most of the time (78.6%), the teacher initiated or co-initiated the stanza. This was to be expected, because one significant role of the teacher in a discourse-based learning environment is to orchestrate participation through social scaffolding (Nathan & Knuth, 2003). However, students also contributed to the social scaffolding role, initiating or co-initiating new stanzas with the teacher about 46% of the time. For example, in Stanza 2, we see the following (note that in this excerpt and others, S refers to student and T to teacher):

T: Well, why don't you go (.) draw yours on the board?
S1: Oh yeah, that's it. (This refers to the previous drawing.)
S2: Ben, draw yours on the board then.

Topics were initiated solely by a student 14% of the time and by an attending researcher 6% of the time.

R Events

R events were coded when a participant responded to an open or closed I event, in keeping with prior conventions (Mehan, 1979; Sacks, Schegloff, & Jefferson, 1974). We observed only two R events in the data set (Stanzas 8 and 28). In each case they followed from known-answer I events that had prompted them (Table 1).

D Events: Analysis of Students' Solution Representations

We look next at the nature of the problem solutions that students offered and the ensuing discussions that took place in response. Students' demonstrations of their solution representations took numerous forms; in fact, we coded 28 unique demonstration representations in our corpus. To convey their ideas when given the opportunity, students made use of a wide array of environmental, graphic, and body-based resources (e.g., Enyedy, 2005). To make sense of the range of solution representations proposed in our data set, we coded each as being in one of four categories: (a) drawings, (b) single objects (e.g., a bowl or tabletop), (c) aggregate objects (e.g., blocks), and (d) representational gestures. For record keeping, representational gestures were coded when students gestured without an accompanying medium such as a drawing or object. We followed Alibali, Heath, and Myers (2001) in distinguishing representational gestures that addressed the content of speech from others, such as beats, that "[did] not present a discernible meaning" (McNeill, 1992, p. 80). Representational gestures are those "in which the hand shape or motion trajectory of the hand or arm represent[s] some object, action, concept or relation" (Alibali & Nathan, 2007, p. 8). The category of representational gesture as used here collapsed the categories of iconic and metaphoric gestures as described by McNeill and others. Representational gestures were coded if solutions were provided through hand and arm shapes without use of any object or drawing.

Drawing representations were produced on the classroom white board using pens. Some involved the use of multiple colors, word labels, speech, or gestures to identify constituent parts. Object representations were coded when students utilized or referred to a prefabricated item. If construction or deconstruction operations involving (or creating) multiple objects were used (including assembly or cutting), then we coded this representation as an aggregate object. If, however, an item was involved only in integrity-preserving transformations (such as rotations or pointing), it was coded as the use of a single object.

Table 2 shows the types of representations along with examples and their frequency. The most common representational forms exhibited by students were drawings (39%) and gestures only (39%). Drawings varied tremendously in how they conveyed in two dimensions the spatial relations of the problem and how they communicated actions and their consequences within this static form. Sometimes labels were used. However, it was more common for students to annotate the drawings with speech and gesture. This annotation was reflected in the extensive use of multimodal forms of communication[2] of the solutions (e.g., Alibali, 2005; Engle, 1998).

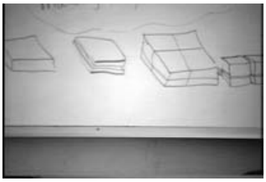
Sometimes students moved away from drawing, relying on more spatially and temporally oriented representational forms. Object use (21%) was the next most common representational form used by students to demonstrate their solutions. In the next section, we report on the interpretations of these representations by participants and the artists themselves. The relative sophistication of students' solution representations is taken up in a later section.

D Events: Analysis of Students' Interpretive Frames

Students' demonstrations conveyed more than their solution methods. These demonstrations also shed light on the interpretive frames that students brought to the

[2]The specific analyses of the multimodal forms of communication are beyond the scope of this article, though they will be the topic of a future article.

TABLE 2
Categories and Frequencies of Representations Used in Demonstration Events, With Examples

Type	*Example*	*n*
Gesture only		11
Drawing		11
Object: Single		3
Object: Aggregate		3
Total		28

problem. The frames, in turn, provided a conceptual point of view from which to analyze students' interpretations of the representations proposed by others.

A *frame* is the enveloping meta-message that conveys the orientation of the participants to the ensuing discourse and thereby influences the interpretation of the actions and utterances (Bateson, 1972). Like speech genres (Bakhtin, 1986), a frame need not be made explicit at the outset. Two prominent and fundamentally different interpretations of *pie* were voiced by the students in this study. The *literal view* of the problem emphasized the properties of a pie: its round structure, the filling, the asymmetry of top and bottom layers of crust (to some, the bottom crust was actually seen as coming up the sides of the pie because both are made from the bottom flat of dough), and so on. The literal frame emphasized the everyday properties of a concrete object, much like the view of cooking gas voiced by the science students in Mortimer and Wertsch's (2003) classroom study. In contrast, the *geometric view* considered the pie to be abstract, used for purposes of mathematical convenience. This frame emphasized the theoretical properties of a hypothetical object, much like the science teacher's view of gas. In Gee's (2004) terminology, it conveys the situated meaning of mathematical practice, in that the exact nature of the pie was arbitrary; it was (or could be) uniform in its composition, and it could even be cast in various geometric forms to suit the demonstration.

Students often (93% of the time) provided solutions that conformed to either the literal (32%) or the geometric (61%) perspective. Table 3 shows examples of D events generated by students that were coded as literal, geometric, and neutral, along with their frequencies of occurrence.

The two contrasting views contributed to the classroom dynamics as students interpreted one another's demonstrations. For example, in Excerpt 1 (for all transcription notations, please see Appendix), we see a student named Dave, who has a geometric stance, offer his solution (see Figure 2):

Excerpt 1 (From Stanza 9)

Dave: Well this is the top ((pointing to the top circle of the pie))
and this is the side ((pointing to the middle portion of the pie
drawing)).
Roger: So you cut through the tin, or you take it out of the tin.

TABLE 3
Demonstration Events ($N = 28$) Coded as Depicting Literal, Geometric, and Neutral Interpretations

Code	*n*	*Comment*
Literal	9	"So you cut through the tin, or you take it out of the tin …"
		"Yeah, yeah … if you have the top crust and you like lift it up, all the stuff's gonna fall out, and what if it's an apple pie. How could you …?"
Geometric	17	"This is just a demonstration of like how you'd see it from that perspective."
		"Well, like before he was saying like … well, then, who's going to eat a pie with just the bottom and I mean just the top and I mean this is just the example. It's just a diagram. I mean nobody's just going to come up here and eat the dry erase board."
		"Here's a pie [*she is holding eight pieces of blocks on her hand*]. It's not round …"
Neutral	2	"They're not equal, they're not equal."
		"Well, I don't really think that the directions were good."

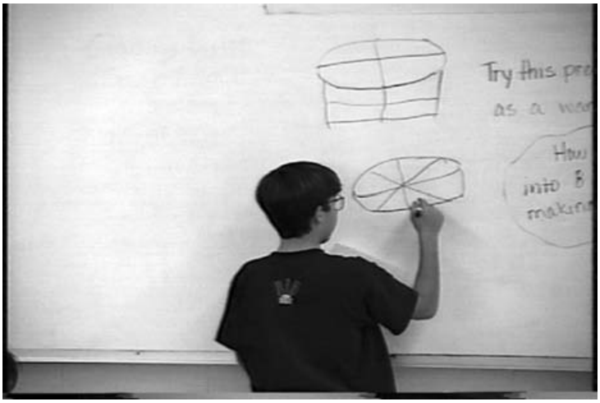


FIGURE 2 A student drawing an example solution representation (from Stanza 9; see Excerpt 1).

Dave:	You took it out of the tin.
S:	(Indecipherable)
Manisha:	So, are you, you're cutting it diagonal here, right?
Dave:	Yeah, ( )
T:	Up or down would be …
Manisha:	So you're doing it now and that won't work, that's not totally equal.
Draper:	I know. There's, they won't. There's, the top's on top and the bottom is on the table. And, and you know that third. You know how here's the pie. If you made a cut to it like that, it wouldn't be the same number of pieces. That line going though the middle …
Manisha:	Yeah.
Draper:	… that separates the two parts, that's not a cut, that's the side of the pie.
Manisha:	Yeah.
Dave:	Well, I know that.
Draper:	So that wouldn't work. (Indecipherable)
T:	Uhhh it's Bob's turn.
Bob:	Like I mean … who would want to have a pie that doesn't have like a bottom dress thing, the thing would like fall off and like I think it's just weird that you cut it through the middle.
S:	This is just a demonstration of like how you'd see it from that perspective.

Dave's solution included cuts that go through the top and sides (Lines 1–3; see Figure 2). Voicing a literal concern, Roger asked about the pie tin (Line 4), and Dave complied (Line 5). Students who had a geometric view had trouble understanding how the cuts (meant to be three-dimensional) are drawn (Line 7) and whether these would actually result in equal-sized pieces (Lines 10–11). Another student, Draper, raised an issue about the interpretation of all of the lines in the drawing (Lines 17–18). Dave agreed (Line 20), but it didn't change his interpretation. Bob raised a different issue: He could not fathom why one would dissect a pie in this manner, as it would lose its integrity. He concluded "it's just weird" (Lines 23–25) and, in a later interaction, challenged the idea because "if you have the top crust and you like lift it up, all the stuff's gonna fall out" (Stanza 37). In defense of Dave's solution, a sixth student took the situated stance that is common in mathematical practice, arguing that the specifics of a pie were inconsequential and were just for purposes of demonstration (Lines 26–27).

Dynamics of the Discourse: Emergence of IDE Sequences and Cycles

In addition to exploring the nature of the events that constitute the discourse, we are interested in the dynamics of the discourse itself. In this section, we investigate the ways in which the discourse institutes change. In subsequent sections, we look at the way forms of IS that pull participants toward or away from shared interpretations interact, and the role these interactions play in sustaining the discourse.

Triadic dialogue—a question, followed by a response, and a subsequent evaluation—is the most common form of discourse pattern observed in classrooms (Lemke, 1990). Our analysis of the pattern of event occurrences revealed that stanzas were often composed of sequences of initiation, demonstration, and evaluation and elaboration. In all, 28 out of the 36 stanzas (77.8%) were made up of IDE sequences. IRE sequences were evident in only 2 of the stanzas (5.5%). Three stanzas were interpreted exclusively as meta-topics, whereas one received both an IRE and a meta-code. Meta-topic stanzas focused on the wording of the Pie Problem itself and followed an alternative structure. The remaining three stanzas did not fit any particular pattern.

The preponderance of IDE sequences was evident not only in the high proportion of total stanzas that followed IDE patterns, as noted above, but also in the total amount of time participants spent interacting within an IDE sequence. The set of 36 stanzas ran 37:30. Of that, 84.3% of the time (31:38) was spent within IDE sequences.

We used the keyword mapping feature within Transana to show the locations of I, D, and E events over the time course of the transcript. Figure 3 reveals the combinatorial nature of the events. The first pattern to observe is that I, D, and E events occurred in sequence quite often (21 times). On only two occasions (Stanzas 2 and 9) did an E event fail to directly follow the referenced D event. In both cases, this occurred when a student interjected another demonstration before students could evaluate the current one. Otherwise, the recurrent pattern

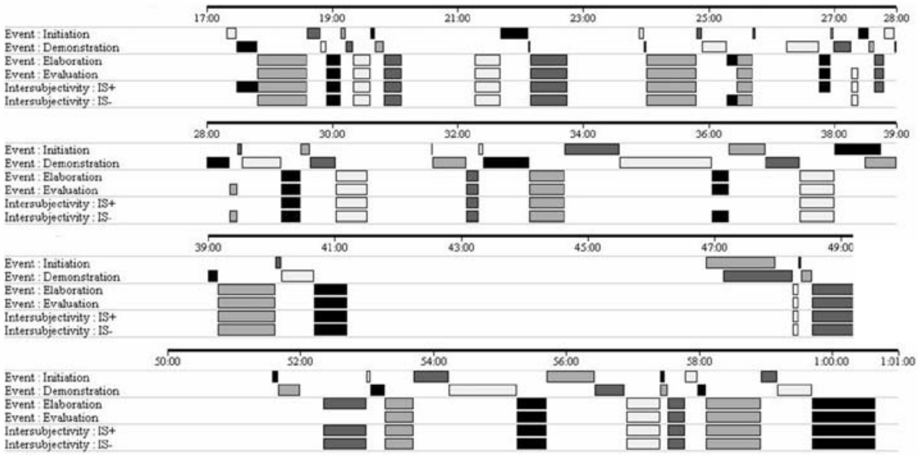


FIGURE 3 The keyword mapping feature in Transana showing the distribution of selected codes along a time line. Grayscale values denote the events (slices in time) to which the codes were assigned. Each of the four time lines shows 11 min of discourse. Initiation, demonstration, and evaluation/elaboration (IDE) events (the first four rows in each time line) tend to follow in sequence. The IDE sequence recurs cyclically. Occurrence of intersubjectivity (bottom two rows of each time line) tends to mark the beginning of the subsequent IDE sequence.

from initiation to demonstration to evaluation/elaboration is seen as one moves along the time line.

The IDE sequence provides a useful way to characterize the discourse structure. Someone, usually the teacher, initiates a new IDE sequence by asking a question or inviting ideas. Participants usually respond to the invitation with some demonstration of their ideas about the solution, most often by making a drawing. Some evaluation ensues, often based on the geometric or literal interpretative frame within which they perceive the demonstration. In IRE patterns, the nature of the evaluation is often perfunctory. In contrast, evaluation of a reply during IDEs reflects what Nystrand and Gamoran referred to as "high level" (Nystrand & Gamoran, 1991, p. 273; Nystrand, Gamoran, Kachur, & Prendergast, 1997), where the reply (i.e., a response or demonstration) to the I event is elevated to the point where the reply modifies the subsequent conversation. This form of high-level evaluation is often accompanied by some elaboration, such as further explanation or even modification of the representation used to convey the solution.

E Events: Establishing IS Among Participants

We also investigated the occurrence of IS as a property of participants' interactions and the ways in which seemingly opposing forces interacted. In this analysis, we drew from the participatory view of IS (Matusov, 1996) reviewed earlier. In this view, as described above, IS$^+$ is cast more broadly than in the traditional view, so that IS$^+$ was considered in evidence not only when students reached agreement, but also whenever students operated within a shared conceptual space. In practice, this often meant that we were evaluating whether we thought interlocutors shared or even appropriated one another's language and representations. IS$^-$ was considered in evidence when students disagreed or presented divergent interpretations. Based on these coding criteria for IS, we achieved perfect interrater reliability ($\kappa = 1.0$). If, in addition to sharing a common representational space, students disagreed, misinterpreted one another, or expressed divergent views, we considered this evidence for both IS$^-$ and IS$^+$.

Instances of IS were ubiquitous in our data set (see Figure 3). All 28 E events—events that exhibited elaboration or evaluation—received at least one of the two IS codes. However, IS codes were applied to none of the I events and to only one D event. IS occurs, by and large, only during the elaboration/evaluation phases of the IDE sequences, as this is where participants assert their level of understanding and agreement with one another.

We also found numerous instances of the co-occurrence of IS$^-$ and IS$^+$ codes. In all, IS$^+$ and IS$^-$ co-occurred in 23 out of 28 E events. Only two cases of IS$^+$ and three cases of IS$^-$ occurred in isolation. In one case where IS$^-$ occurred without any accompanying IS$^+$, a student created a folded paper model that, when cut, was to yield eight pieces. This model had the potential to convey many difficult aspects of the solution, such as its three dimensionality and the creation of multiple pieces over time. However, the student had added additional folds so that more than eight pieces were produced (another student commented that it looked like "about 20"). She also had difficulty handling the pieces and scissors and dropped the pieces partway through her cutting demonstration. Participants voiced disagreement with her conclusions and confusion about her method, earning an IS$^-$ code. In the end, no one picked up on the demonstration in subsequent speech acts.

As expected from the participatory view of IS, seemingly conflicting aspects of IS commingled to create rich discourse, as evidenced in Excerpt 2 (see Figure 4):

Excerpt 2 (From Stanza 20)

Bob: ((Drawing 2 circles one above the other.)) Alright here is the
top and here is the bottom. Just say that they're like if you really
look at it like, and like if you cut it like this
Manisha: Then it would have to go all the way through.
S2: Yeah, it would have to go all the way through. Right here is like
… wait is this like the bottom?

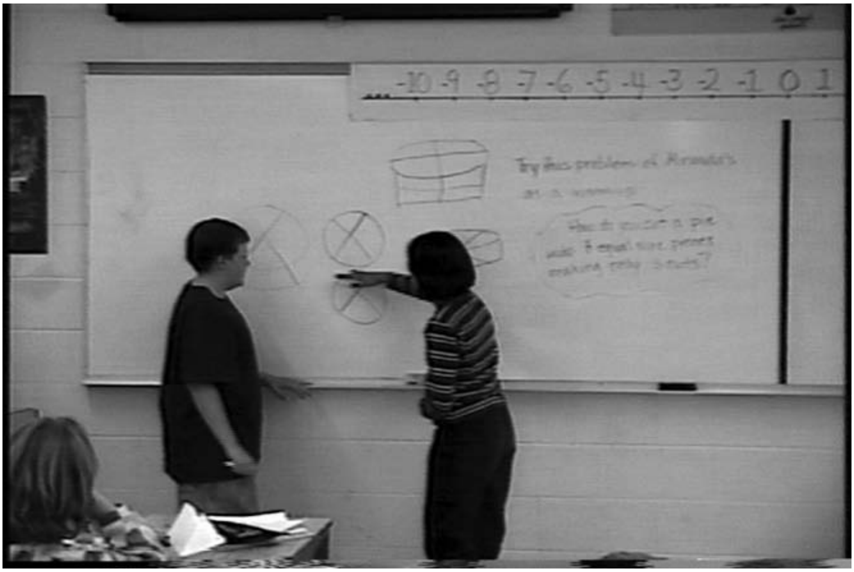


FIGURE 4 Students discuss the interpretation of a drawing that shows two separated layers of the pie that are then each cut twice (from Stanza 20; see Excerpt 2).

7	S3:	That's the side.
8	Manisha:	And then you'd have to cut it in half (using Bob's drawing as
9		reference).
10	S:	(Indecipherable)
11	Researcher:	Can you guys speak up?
12	T:	Guys you need to talk louder okay?
13	Bob:	You cut all the way down. That wouldn't make eight pieces.
14	Manisha:	If you cut it in half it would.

In Excerpt 2, Bob provided a drawing (Lines 1–3) that showed two separated layers of the pie (one top view of a circle drawn above another; see Figure 4). Although this captured the three-dimensional aspect of the problem for some students, to Bob, it reinforced his assertion that there were four rather than eight pieces made (to Bob, the four shown on top and again on the bottom were the same pieces; Line 13). Manisha pointed to the board to address the representation, thereby adopting Bob's representational frame. This indicated a shared discourse space characteristic of IS^+. However, she also conveyed an alternative interpretation, arguing that the cuts shown in each layer had to "go all the way through" from one layer to the other (Line 4). She stood and actually physically entered Bob's space at the board and reframed his drawing (Lines 8–9) while using her hand gestures to refer to the space between the two layers that, in her mind, captured the third and crucial cut that yielded eight (rather than four) pieces. Bob challenged her interpretation (Line 13), but Manisha reasserted her initial point (Line 14) that if someone cut all four pieces in half horizontally, it would yield eight equal pieces.

The frequent co-occurrence of IS^+ with IS^- during E events is in keeping with the participatory view that maintains that these two forms of IS are not mutually exclusive within the discourse and may even be necessary for substantive disagreements to occur.

Sustaining of the Discourse: IDE Cycles and the Role of IS

One further insight from the keyword map of Figure 3 is that IDE sequences tend to be cyclical, chaining one to another, thus constituting a "dialogic" interaction (Lotman, 1988; Wells & Arauz, 2006). Most (81%) IDE sequences followed directly from a previous IDE sequence (ignoring the first stanza and the stanza immediately following the classroom break in this calculation). In contrast, we observed no chaining of IRE sequences, a finding reported elsewhere (Wells & Arauz, 2006). Chaining is seen in the IRF (follow-up question) variant (Wells & Arauz, 2006), but it was not observed in this corpus.

Students had many sources of difficulty interpreting one another's (and their own!) line drawings. Several IS^- instances were the apparent result of inadequate drawing as students tried to convey their three-dimensional ideas using two-dimensional drawings on the white board. For example, Dave's drawing in Figure 2 showed three lines all intersecting in a way that violates principles of perspective (e.g., foreshortening).

Several other points of conflict centered on the interpretation of the curved, convex edge (see Figure 2 and Excerpt 1) that is formed through use of perspective when the flat, top layer intersects the curved side (Waltz, 1975). In some cases, students (even the artists themselves) would (re-)interpret the curved line as a cut when they counted the resulting pieces:

Excerpt 3 (From Stanza 1)

1	Manisha:	That makes … four pieces on the bottom and four pieces on top.
2	Bob:	What's wrong with mine?
3	S:	Yours is, you didn't circle.
4	Bob:	Who cares?
5	Roger:	That's six pieces.
6	Bob:	Fine.
7	S:	No!

In Excerpt 3 (Stanza 1), a student with a literal view actually used the individually bounded regions shown on the board to count the number of pieces that resulted from making the cuts and concluded, “That’s six pieces” (Excerpt 3, Line 5). This conclusion conflicted with the geometric interpretation offered by another student that led to eight pieces when interpreted in three dimensions (Line 1).

Although IS codes were common throughout this discussion, they appeared in key places in the discourse. All of the 26 IDE sequences that followed a prior stanza (i.e., excluding the first one and the one following the classroom break) were preceded by statements coded for IS. As noted, these were almost exclusively found during E events. Most of these IDEs (85%) showed the co-occurrence of IS^+ and IS^-. Furthermore, IS was located in E events that preceded all but three IDE sequences.

The role of IS, then, appeared to be substantial in this extended classroom discussion. In its convergent or positive form, it identifies the common discursive space within which participants can meet about their problem-solving ideas. In its divergent or negative form, it reflects the challenges that speakers face in communicating to others, particularly when the representations are ambiguous and when interlocutors have differing interpretive frames and limited skill with three-dimensional drawing. Furthermore, in its general form, IS marks further triadic event sequences that structure the group problem-solving interactions. Thus, within the context of the high-level (Nystrand & Gamoran, 1991) E events, IS designated and may have contributed to the cycling of the IDE event sequences that were so prevalent in these data.

Structure of the Discourse

From these meso-level (event-level) analyses, we concluded that the discourse was largely structured by IDE sequences. IDE became central to our study because I events tended to mark the boundaries of discourse units (stanzas) and then served to elicit through open-ended invitations rich responses from students in the form of representation-laden demonstrations (D events). It is within D events that students often showed their mathematical thinking. D events, in turn, triggered E events, in which the proposed solutions were evaluated and elaborated upon, often by new speakers. It is within the IDE triads that discourse participation is enabled and supported.

The analyses showed that the recurrence of IDE sequences was preceded by, and perhaps even driven by, IS—through disagreement, inadequate representational skills, and conflicting interpretations among members of the class (IS^-) on the one hand, and through agreement, elaboration, and use of shared representations (IS^+) on the other. Regular prompting by the teacher and students served as a catalyst for students’ engagement with the task and with one another. But the discourse was actually sustained through IDE cycles reflecting students’ engaged pursuit of ideas and solution representations. However, at this level of analysis, we can only describe this process as sequences of events. To see the landscape that this discussion actually traverses, we need to step back and examine the discourse from a global level.

Direction of the Discourse: Trends Toward Representation Standardization

The preceding findings paint a portrait of the discourse as seen at the event level. We see the particulars of students' interactions and the frequencies of their occurrence. However, as with a pointillist painting viewed up close, it is difficult to determine the nature of the picture. Thus, we pull back a bit so we can look at the discourse at a more coarsely grained level. At the macro level of analysis, we are principally interested in the terrain the discourse covers. In other words, where does it start and where does it go? To assay whether the interactions were productive, we examined changes in the discourse, as speakers received feedback and monitored their own views.

Demonstrations of solution representations operate as objects (sometimes literally) around which IDE sequences organize. One way to characterize the discourse is to examine the evolution of the uses of representations over time. Toward this end, we examined the first and last uses of representations by one of the students—in this case, one with a literal view of the Pie Problem. Bob was a fairly vocal member during this session, although he was generally not a central participant in math class and typically scored near the bottom of his class. He appeared to be very engaged in this lesson, however, and provided some of the earliest as well as some of the latest contributions to the discussion. For this reason, he makes for an important case. Later, we show the changes across all of the representations used over the entire discourse.

Figure 5 shows Bob's first demonstration (Stanza 3). In addition to the drawing he made, Bob used verbal explanation and gestures to elaborate his intentions, as shown in Excerpt 4:

Excerpt 4 (From Stanza 3)

Bob: OK, here's the pie (sigh).
T: Let's listen to Bob now please.
S: I drew them like square bodies … like that. And they're curving …
Bob: Well, fine rou:::nd.
Bob: Yeah, Dude. It does … cuz you're cutting it in half.
S: No.
Mary: No … No … you're, it usually goes to the bottom (Indecipherable) pie, … usually goes to the bottom.

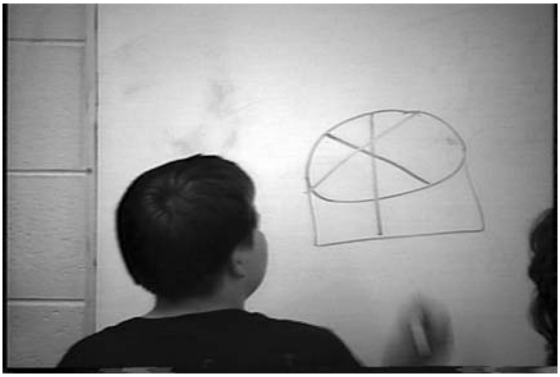


FIGURE 5 Early in the discussion (see Stanza 3, Excerpt 4), Bob demonstrates his drawn solution to the Pie Problem.

T2:	Bob, Bob how many slices are there?
Bob:	What?
T2:	How many pieces are there?
Roger:	There's 12.
Mary:	((Counting each piece))
Bob:	No, not there ((touching the extraneous middle triangle)).
Bob & Mary:	((Pointing inside each bounded region of the drawing while counting)) One, two, three, four, five, six, seven, eight.
S:	No.
T2:	No, no, no, no, no.
Mary:	((Pointing at each region while counting)) One, two, three, four … eight. No.
Mary:	((Erases Bob's drawing))
S:	Yeah, but they're not equal.

The drawing was of a pie (Excerpt 4, Line 1), and Bob completed its three-dimensional depiction before he added the lines that represented the cuts. The pie was to be cut up by three straight lines: The first was a completely straight vertical line through the top ellipse and the side, whereas the other two lines formed an *X*. It was the artist's intention that the three cuts meet in the middle, and when they formed an unintended triangular shape in the center of the ellipse, it was discounted (Line 14). To Bob, and another student up at the board, it was natural to equate the separate regions with pieces, and so Bob and Mary each counted to eight, pointing and touching each region as they went (Lines 15–16). Not all participants agreed with this (Lines 17–18), but a second count by Mary, who was skeptical of the solution, lent further support to the view that there were eight pieces, given the region-as-pieces interpretation (Lines 19–20). Still, Mary was unwilling to believe her count and she concluded "No" and erased the drawing (Lines 20–21).

Bob's initial demonstration can be evaluated with respect to three criteria:

1. *External consistency:* The adherence of the demonstration to the principles of perspective drawing;
2. *Internal consistency:* The uniformity with which elements of the representation take on certain meaning or roles in the solution; and
3. *Ambiguity:* The amount of effort and elaboration needed to interpret the drawing in an unambiguous manner.

On the first criterion, the representation violated some of the principles of perspective drawing. The diagram did not convey a sense of depth or of a vanishing point (often referred to as *linear perspective*) and lacked proper size and shape variation. Thus, we see the top plane of the pie from the top view (as fairly circular) but the sides from a side view. The drawing also did not apply notions of modeling that would, for example, lead one to expect the vertical slice to bend at the side of the pie as it changed planes. On the second criterion, we see inconsistencies within the drawing itself. The artist misinterpreted the curved, convex edge, as was evident by his counting method. Finally, there were aspects that were ambiguous (such as the addition of the triangular region and the square edges) and required further explanation from the student.

Excerpt 5 (From Stanza 33)

Bob: Can I go up?
T: Bob, since people seem to be directing at you, Bob, I think it's only fair you have a chance to speak out.
Bob: Okay ((walking up to the board with a hand full of blocks)).
T: We're gonna spend five more minutes on this and then we have to move on. And we can come back to it, but for today, five more minutes.
Bob: Okay, um … ((places 8 blocks into his right hand in a cube formation))
Bob: Okay, say I'm … this is a pie, and you cut it like right there ((using hand like knife makes a cutting motion to top of cube perpendicular to his chest)) and right there ((using hand like knife makes a cutting motion to top of cube parallel to his chest)). And then you cut it at the bottom ((using flat hand, palm up, like cutting the cube in two layers)). That is still going to be four pieces because you cut it at the bottom.

In contrast to his early solution, Bob's final demonstration (Excerpt 5; also see Figure 6) used eight wooden cubic blocks to form a $2 \times 2 \times 2$ cube. He intended for the cube to be the pie (Line 10) and his hand motions to represent the slices (Lines

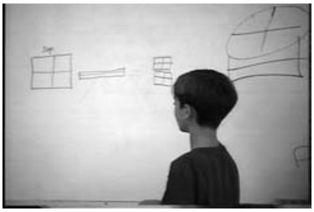
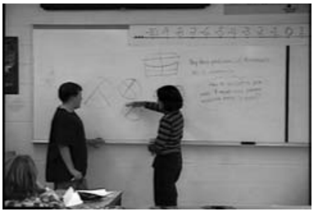
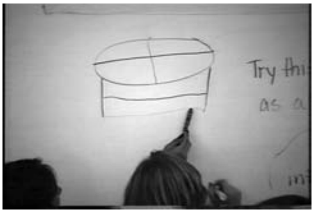
FIGURE 6 Near the end of the discussion (Stanza 33, Excerpt 5), Bob demonstrates a solution to the Pie Problem using aggregate objects (blocks).

10–15). He used this representational form to argue that the horizontal layer cut (Lines 14–15) was not legitimate because the horizontal cut was "at the bottom" (Line 14). To those in the geometric camp, this would have been the side of the pie. To Bob, the horizontal cut did not really result in doubling the number of pieces that one would expect to get when eating a real slice of pie with both a top and bottom crust, so there were only four pieces of pie (Line 15–16).

Bob took issue with the geometric interpretation that all cuts are reasonable and that all cuts make more pieces. He did not abandon the literal view that he had initially. However, his use of representation showed a rather significant shift. It was more refined, more conventional, and more explicit, suggesting he was more aware of the perspectival needs of others (Greeno & MacWhinney, 2006; Greeno & van de Sande, 2007). The three-dimensional object was consistent with principles of perspective (Criterion 1), and the components of the representation maintained a stable meaning throughout his demonstration (Criterion 2). Furthermore, the use of an object with gestures showed the spatial and temporal aspects of his solution in an explicit manner (Criterion 3). Although one may have disagreed with his interpretation of the layer cut, his position was consistent and relatively unambiguous.

The changing nature of representation use seen here gives a sense of the global dynamics of the discourse. To place these contrasting demonstrations within the larger corpus, we applied a four-level rubric based on the three criteria above (external consistency, internal consistency, and ambiguity) to the entire discourse (interrater reliability produced 83% agreement and $\kappa = 0.77$ across the levels). As Table 4 shows, we identified 46 representations across all of the D and E events over the 36 stanzas. Level 1 identified the most impoverished representations, and they were observed with the greatest frequency. Levels 3 and 4, the most refined

TABLE 4
Frequency, Description, and Visual Examples of Each Coded Level of Standardization Representations in Both Demonstration and Evaluation/Elaboration Events ($N = 46$)

Level	*n*	*Description*	*Example*
4	5	• The representation is completely consistent with the principles of perspective. • The meaning or role of each component of a representation is applied uniformly. • Ambiguity of interpretation is kept to a minimum, and resolving it requires little effort or elaboration.	
3	6	• The representation is somewhat consistent with the principles of perspective, though there is one violation or omission. • The meaning or role of a minor component of a representation is not applied uniformly. • There is some ambiguity of interpretation, but it is resolved with a small amount of conscious effort and/or elaboration.	
2	10	• There is an attempt to be consistent with the principles of perspective, though there is more than one violation. • The meaning or role of an important component of a representation is not applied uniformly. • There is a great deal of ambiguity of interpretation that is resolved with a large amount of conscious effort and/or elaboration.	
1	25	• The representation disregards the principles of perspective. • The meaning or role of an important component of a representation is not applied uniformly. • Interpretation of the representation is highly ambiguous and requires a substantial effort and/or elaboration.	
Total	46		

representations, were relatively rare. Figure 7 shows the distribution of the level codes over time. Rows represent the four levels from the rubric. Changes in grayscale designate video clips that received codes, with unique grayscale value assigned to each clip and the length of a clip proportional to its duration. As Figure 7 shows, codes for the more idiosyncratic and ambiguous representations (Levels 1 and 2) were scattered throughout the discourse, though they appeared to be in more frequent use early on. Often, these were casual drawings that paid little attention to accuracy or conventions of perspective. Most were representational gestures and drawings that demonstrated spatial and temporal relations in an idiosyncratic and ambiguous manner and that addressed only part of the solution. Later in the discourse, it was far more common for students to propose solutions that were more standardized (Levels 3 and 4). For example, in the second half of the discourse, students introduced objects that they could manipulate in space to show three-dimensional relationships that inherently addressed issues of perspective; some actually yielded eight equal-sized pieces. In the latter half of the discourse, we also observed more frequent use of drawings from multiple viewpoints with verbal labels to address the limits of the two-dimensional medium (e.g., see the example in Table 4 for Level 4).

It appeared that the quality of the representations, as gauged by our three criteria, actually improved over time. As a statistical check that this pattern was in evidence, we split the entire transcript in half and compared the coded levels for the first half of the discourse to those for the second half. We found that it was reliably more likely for students' representations to receive higher levels in the second half of the discourse, $t(40) = 3.27$, $MS = .35$, $p < .005$. Indeed, Level 4 representations ($n = 5$) were observed only in the second half of the discourse.

This change in the use of representations appeared to be motivated by earlier interactions and, perhaps, a sense that one could be more persuasive with the right kind of demonstration. Early on, students were willing to use solution representations of any sort. They invented them readily and executed them casually. But as the discourse continued without resolution, students became more refined in their methods and drew more often on standard ways of depicting the problem and the important mathematical relationships. This was akin to Enyedy's (2005) finding

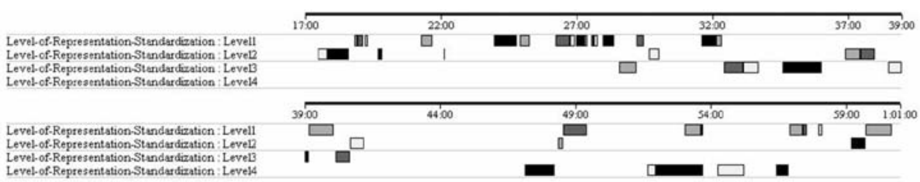


FIGURE 7 Transana keyword map showing the distribution of the four representational levels as assigned to each solution representation ($N = 46$) over the time course of the discourse. Each time line spans 22 min.

among younger children who invented maps that incorporated topological information. In that study, social interactions of co-authorship and collective transformation led students to establish conventions that received greater consensus.

As we look across time in Figure 7, we can see that the discourse took a discernible direction even though there was no central agent or explicitly stated goal guiding it. As explored earlier at the event level, IS clearly served an important role in perpetuating the discourse. Here at the more global level, we begin to see how participants developed their uses of representations in service of IS. In this way, we see how knowledge accumulates socially, mediated by the interactions of the whole-classroom discourse.

DISCUSSION

This article identifies several elements that appear to be central for advancing researchers' understanding of socially mediated learning. We have seen how the classroom discourse under investigation naturally decomposed into a set of smaller discussions around key occurrences or ideas—stanzas, in Gee's (2005) terminology—and how it tended to move from stanza to stanza as students' ideas and their understandings of one another's viewpoints changed. At the global level, for example, we noted important shifts in the nature of the representations used by students to articulate their positions.

Stanzas themselves exhibited an internal structure—event sequences, in our terminology—that were specific to these circumstances. In traditional classrooms, the IRE triad is a direct manifestation of a particular social relationship between the students and a central authoritative figure, and it shows little self-perpetuation. The specific structure of events differed in this discourse-centered classroom. As we saw, chains of IDE sequences emerged as the underlying structure that provided the rhythm of this discourse. (In fact, the teacher had to demand that the discussion of the Pie Problem end so other class matters could be attended to.)

Although the importance of IS in social interaction is well documented, its role in shaping discourse and socially mediated learning is only beginning to be understood. We found that IS played a central role. At the event level, it occurred almost exclusively during the high-level E events that combined evaluation and elaboration. It also consistently marked new IDE triads. In this way IS designated, and may have even helped to bring about, the preconditions for the protracted dialogic interactions that ensued.

Challenges to a convergent solution were largely attributed to (a) fundamentally different interpretative frames of the problem and solution representations exhibited by those with literal and geometric views, and (b) inadequacies within the representations themselves. Still, IS was evident through the establishment of a common basis for communication even when convergence to a common solution or representation was clearly blocked. Over time, the demonstrations evolved from relatively casual and ambiguous representations to representations that were more principled and explicit about the spatial and temporal relations of the proposed solutions. This shift to more widely accessible representations is the kind of change one expects among speakers striving for a shared understanding.

These analyses revealed aspects of the structure of the discourse as well as its dynamics, as class members used communication and representation to explore the Pie Problem. Communication and representation are central aims of reform mathematics instruction in their own rights, expanding the range of mathematical competencies beyond calculation and fact retrieval (National Council of Teachers of Mathematics, 2000). Classroom discourse provides other benefits as well. Teachers gain great insights when students share their thinking. Student discourse informs teachers' assessments of students' current thoughts and also contributes to longer term models of conceptual development (Nathan, Elliott, Knuth, & French, 1997). For students to participate and be engaged, they must share a great deal. However, teachers working to adopt reform practices also acknowledge that they must conduct their classes in new ways, and this places large and unfamiliar demands on them, as they must also monitor students' participation with respect to curricular and administrative objectives. Herein lies one of the greatest challenges of managing the discourse-oriented classroom. In the class studied here, the teacher attended to many things in her design of the learning environment, including the establishment of classroom norms for listening and presenting, mathematical notation, the vocabulary of problem-solving strategies, and so on. The set of shared knowledge and practices is never bounded, however, and each activity creates new challenges for manifesting engaging and transformative discussion.

When sustained discourse occurs, the perceptive instructor tries to cultivate it and give it space, circumstances permitting. So it was with the instructor of the class we observed. The teacher played a significant role, but one that could be described as catalytic rather than central. Her efforts were directed mainly at social scaffolding of the discourse (Nathan & Knuth, 2003; Yackel & Cobb, 1996). Student talk clearly dominated the room. The presentations of mathematical ideas and their evaluations were largely left up to the student participants. What developed was a healthy, sustained mathematical discourse. Students posed solutions, asked questions, critiqued one another, and reformulated ideas in hopes that the next round would be better—more accurate, more widely understood, and more persuasive. Indeed, students' desires to make themselves understood and convincing appeared to play a critical role in the dynamics of this discourse.

In their longitudinal analysis of mathematics classroom discourse, Nathan and Knuth (2003) defined *productive discourse* as "forms of social exchange which provide participants with an avenue to construct and build upon mathematically correct conceptions through their interactions with other class members" (p. 204). In reviewing the current classroom interaction, we see several indicators that lead us to believe that this was a productive discourse. First, participants worked together, continually reflecting on one another's ideas. In this sense, students were engaged in a *recursive communication process*: They listened to one another and were genuinely interested in other's ideas and contributions (Rommetveit, 1989). Students built on other's ideas, even when they did not agree. Moreover, in several cases, students appropriated one another's representations. This finding is reminiscent of Latour's (1996) notion of *interobjectivity*, in which people with divergent points of view can still exhibit coordinated interaction through the use of shared objects and representations.

Students also evaluated and reflected on the activity itself. For example, they found ways through peer-to-peer interactions to bridge theoretical and practical speech genres that they may have otherwise resisted had these efforts come from the teacher (cf. Mortimer & Wertsch, 2003). There were several meta-stanzas in which students commented on the nature of the problem statement and offered ways to reword it in accordance with their own understanding. These occurred only after protracted discussion of the solutions and the different interpretive frames.

Throughout the discourse, students disagreed with and challenged one another, but did so respectfully and productively, pushing peers to be clearer and more mindful of different interpretations. And in the end, the discourse did not seem to convert many students from their initial interpretations to new ones. Rather, disagreements during the discourse spawned clarifications and standardization of solution representations. In this way, the disagreements fostered critical dialogue (Bakhtin, 1990) and led students to articulate their disparate positions in more sophisticated ways.

While these analyses reveal the influences of IS, IS does not tell the whole story about sustained and productive classroom discourse. For if IS is the objective in constructivist classrooms, then representation is the means by which it is to be achieved. By working across interpretive frames, those with different views have to refine their ideas, forms of communication, and representations, in much the same way scientists do when working across traditional disciplinary boundaries (Hall, Stevens, & Torralba, 2002). It is also through the presentation of one's ideas that acceptance and substantive disagreement can occur. As the students in this classroom revealed, such acceptance and disagreement are not easily achieved through casual presentations. Too often, idiosyncratic representations meant little to the audience, and they were sometimes even misinterpreted by the artists themselves, as Draper pointed out in Excerpt 1 (Line 15; also see Figure 2), in which Dave took the curved edge to be a drawn cut. Instead, students came to adopt representational forms that carried common meanings that were assayed with minimal effort. This finding is similar to Schwartz's (1995) insights about the solutions generated by students operating in dyads. Schwartz found that working across multiple representations—multiple frames, if you will—was a demand unique to those students working in teams. The dyadic process required more communication than that of students working on their own, and it fostered production of more abstract representations, which contributed to dyads' superior problem-solving performance. Similarly, in the study reported here, students adjusted their demonstrations to suit the group's needs.

It is reasonable to ask if the choice of the Pie Problem was instrumental in bringing about this interaction. Of course, with these limited data, we cannot know for certain. But, if one is seeking a central cause, we think that the better bet is with a classroom environment and norms of interaction that enable students to be reasonably secure in their efforts to share their ideas and to critique one another's ideas and solution proposals. In terms of the generality of the findings, we would expect this classroom structure to support the kind of IDE chaining we observed over a wide range of activities before we would expect the Pie Problem to support IDE in a more restrictive environment.

The study of IS is growing in importance as the bounds of cognitive science expand (Stahl, 2006). There is greater awareness of the essentially social nature of human thought and learning, as well as a growing appreciation of the complexities of designing and managing socially mediated learning environments. Within the maturing fields of embodied cognition and cognitive neuroscience, basic interpersonal processes such as imitation, empathy, and the ability to impute the intentions of others—all behaviors that hinge on IS—are being considered vital to advancing researchers' understanding of fundamental mechanisms of both individual and social behavior.

Researchers studying imitation and the comprehension of observed actions and emotional facial expressions have found that participants (primates, in many of the studies) necessarily engage their own motoric and emotional processes through the system of mirror neurons when they observe the actions and feelings of others (Rizzolati et al., 2001). Currently, the claim is that mirror neurons are specially evolved and selected areas of brain circuitry "that allow us to appreciate, experience and understand the actions we observe, the emotions and the sensations we take others to experience" (Gallese, 2003a, p. 525) by constituting them in intersubjective relation to our own actions and feelings. This process is called *embodied simulation*. More recently, investigators are drawing on embodied simulation to explain behaviors that are considered more complex than imitation and comprehension of actions; it can, for example, also explain findings on the comprehension of emotionally laden text (Havas, Glenberg, & Rink, 2007).

The embodied simulation model of empathy and imitation does not necessitate that participants experience others' actions and emotions in the same way that they experience their own. Rather, the model stresses that to appreciate others, participants must share a common interpersonal space—the manifold of intersubjectivity—within which the embodied simulations operate and facilitate participants' interpretations of the world around them (Gallese, 2003a, 2003b). This notion is consistent with both the participatory and the radical constructivist views of IS reviewed earlier (Steffe & Thompson, 2000). Although large gaps between neuroscience and educational practices still persist (Bruer, 1997, 2006), theoretical and empirical advances in the study of empathy, imitation, and embodied cognition contribute to researchers' appreciation of the role that social factors play in shaping individual behavior and perhaps even basic aspects of cognitive architecture. As investigators' understanding of socially mediated learning and communication continues to develop, they can expect to see a greater exchange among these previously disparate fields of inquiry.

Students engaged in collaborative problem solving and substantive mathematical argument are a sight to behold, and such a collaborative learning process has become one of the critical markers of successful reform-based classrooms (e.g., Nathan & Knuth, 2003; Strom, Kemeny, Lehrer, & Forman, 2001). As teachers permit—and even invite—students to publicly share their multiple perspectives, the need to understand more completely the nature and dynamics of IS increases. As Bakhtin (1990) pointed out, dialogic exchanges of this form are necessary; without them, we learn nothing, and do little to advance and refine our understanding and our means of communicating our understandings to others.

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APPENDIX
Transcription Excerpts With Jeffersonian Notation
Transcription Conventions

[	Point of overlap onset
]	Point of overlap termination
=	No interval between adjacent two turns
(2.3)	Interval between utterances (in seconds)
(.)	Very short untimed pause
<u>word</u>	Speaker emphasis
the:::	Lengthening of the preceding sound
?	Rising intonation, not necessarily a question

,	Low-rising intonation, suggesting continuation
.	Falling (final) intonation
CAPITALS	Especially loud sounds relative to surrounding talk
° °	Utterances between degree signs are noticeably quieter than surrounding talk
↑ ↓	Marked shifts into higher or lower pitch in the utterance following the arrow
( )	A stretch of unclear or unintelligible speech
(( ))	Nonverbal actions

Excerpt 1 (From Stanza 9)

Dave: Well this is the top ((pointing to the top circle of the pie)) and this is the side ((pointing to the middle portion of the pie drawing)).
Roger: So you cut through the tin, or you take it out of the tin.
Dave: You took it out of the tin.
S: (Indecipherable)
Manisha: So, are you, you're cutting it diagonal here, right?
Dave: Yeah, (   )
T: Up or down would be …
Manisha: So you're doing it now and that won't work, that's not totally equal.
Draper: I know. There's, they won't. There's, the top's on top and the bottom is on the table. And, and you know that third. You know how here's the pie. If you made a cut to it like that, it wouldn't be the same number of pieces. That line going though the middle …
Manisha: Yeah.
Draper: … that separates the two parts, that's not a cut, that's the side of the pie.
Manisha: Yeah.
Dave: Well, I know that.
Draper: So that wouldn't work. (Indecipherable)
T: Uhhh it's Bob's turn.
Bob: Like I mean … who would want to have a pie that doesn't have like a bottom dress thing, the thing would like fall off and like I think it's just weird that you cut it through the middle.
S: This is just a demonstration of like how you'd see it from that perspective.

Excerpt 2 (From Stanza 20)

Bob: ((Drawing 2 circles one above the other.)) Alright here is the top and here is the bottom. Just say that they're like if you really

look at it like, and like if you cut it like this
Manisha: Then it would have to go all the way through.
S2: Yeah, it would have to go all the way through. Right here is like … wait is this like the bottom?
S3: That's the side.
Manisha: And then you'd have to cut it in half (using Bob's drawing as reference).
S: (Indecipherable)
Researcher: Can you guys speak up?
T: Guys you need to talk louder okay?
Bob: You cut all the way down. That wouldn't make eight pieces.
Manisha: If you cut it in half it would.

Excerpt 3 (From Stanza 1)

Manisha: That makes … four pieces on the bottom and four pieces on top.
Bob: What's wrong with mine?
S: Yours is, you didn't circle.
Bob: Who cares?
Roger: That's six pieces.
Bob: Fine.
S: No!

Excerpt 4 (From Stanza 3)

Bob: OK, here's the pie (sigh).
T: Let's listen to Bob now please.
S: I drew them like square bodies … like that. And they're curving …
Bob: Well, fine rou:::nd.
Bob: Yeah, Dude. It does … cuz you're cutting it in half.
S: No.
Mary: No … No … you're, it usually goes to the bottom (Indecipherable) pie, … usually goes to the bottom.
T2: Bob, Bob how many slices are there?
Bob: What?
T2: How many pieces are there?
Roger: There's 12.
Mary: ((Counting each piece))
Bob: No, not there ((touching the extraneous middle triangle)).
Bob & Mary: ((Pointing inside each bounded region of the drawing while counting)) One, two, three, four, five, six, seven, eight.
S: No.
T2: No, no, no, no, no.
Mary: ((Pointing at each region while counting)) One, two, three, four … eight. No.

Mary: ((Erases Bob's drawing))
S: Yeah, but they're not equal.

Excerpt 5 (From Stanza 33)

Bob: Can I go up?
T: Bob, since people seem to be directing at you, Bob, I think it's only fair you have a chance to speak out.
Bob: Okay ((walking up to the board with a hand full of blocks)).
T: We're gonna spend five more minutes on this and then we have to move on. And we can come back to it, but for today, five more minutes.
Bob: Okay, um … ((places 8 blocks into his right hand in a cube formation))
Bob: Okay, say I'm … this is a pie, and you cut it like right there ((using hand like knife makes a cutting motion to top of cube perpendicular to his chest)) and right there ((using hand like knife makes a cutting motion to top of cube parallel to his chest)). And then you cut it at the bottom ((using flat hand, palm up, like cutting the cube in two layers)). That is still going to be four pieces because you cut it at the bottom.