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ORIGINAL ARTICLE

Representational disfluency in algebra: evidence from student gestures and speech

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Abstract In this investigation, we analyzed US middle school students' (grades 6–8) gestures and speech during interviews to understand students' reasoning while interpreting quantitative patterns represented by Cartesian graphs. We studied students' *representational fluency*, defined as their abilities to work within and translate among representations. While students translated across representations to address task demands, they also translated to a different representation when reaching an impasse, where the initial representation could not be used to answer a task. During these impasse events, which we call *representational disfluencies*, three categories of behavior were observed. Some students perceived the graph to be bounded by its physical and numerical limits, and these students were categorized as *physically grounded*. A second, related, disfluency was categorized as *spatially grounded*. Students who were classified as spatially grounded exhibited a bounded view of the graph that limited their ability to make far predictions until they physically altered the spatial configuration of the graph by rescaling or extending the axes. Finally, students who recovered from one or more of these disfluencies by translating the quantitative information to alternative but equivalent representations (i.e., exhibiting representational fluency), while retaining the connection back to the linear pattern as graphed, were categorized as *interpretatively grounded*. Understanding the causes and varieties of representational fluency and disfluency contributes directly to our understanding of mathematics knowledge, learning and adaptive forms of reasoning. These findings also provide implications for mathematics instruction and assessment.

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1 Introduction

Representational fluency, the ability to work within and translate among representations, is central to the enterprise of mathematical activity and knowledge construction. In this investigation, we analyzed students' gestures in relation to their speech, writing, and drawing during interviews to understand the reasoning processes and strategies children use while interpreting Cartesian graphs of quantitative patterns. In framing our research, we draw on prior work on reasoning with graphs, as well as the role of representational fluency more generally in shaping students' reasoning processes. Central to our approach is how children assign meaning to formal, mathematical representations—how they attempt to ground them in terms of actions on these representations and with reference to other mathematical formalisms. When analyzing students' responses to pattern generalization tasks involving Cartesian graphs, we found students exhibiting *representational disfluencies*, events where perceived shortcomings in using representations to solve problems motivate students to modify or translate among representations. Close analyses of students' speech, gestures, and writing revealed several new insights into the ways mathematical representations are perceived and interpreted by students, namely how *bounded* students are to the physical limits of a given graph and how such boundedness suggests ways that students ground their understanding of the task demands in a given representation. These insights have direct implications for theories of the nature of mathematical reasoning with formal representations, as well as for the role that careful observations of students can play for informing instruction.

1.1 Gesture research

The spontaneous hand gestures that a child produces during problem solving often reveal the way the child represents the task (Goldin-Meadow, 2003). In particular, students' gestures, combined with their words, serve as semiotic ways that students construct meaning of the representations they encounter, as mediated by students' knowledge and perceptual processing (Radford, 2009). Roth (2001) reviewed evidence that gestures reveal implicit or emergent knowledge that may be expressed in speech only at some later point (Church & Goldin-Meadow, 1986; Crowder, 1996). During explanations of problem solving, attending to one's gestures "can be used to make inferences about precisely which features of the complex perceptual field … should be attended to" (Goodwin, 1994, p. 614).

The relationship between speech and gesture is significant. McNeill (1992) argued that gesture and speech constitute two components of "a single underlying mental process" (p. 1) that mediates language production, even though gestures do not necessarily reveal identical information. By tracking gestures, one can learn not only what someone is thinking about, but also the *manner* in which a speaker conveys those thoughts. It is in the sense that gestures provide "the imagery of language" (McNeill, 1992, p. 1). Gestures can also reveal how a speaker orients to and incorporates objects (both physical and symbolic) from her immediate environment into her discourse.

1.2 Grounding

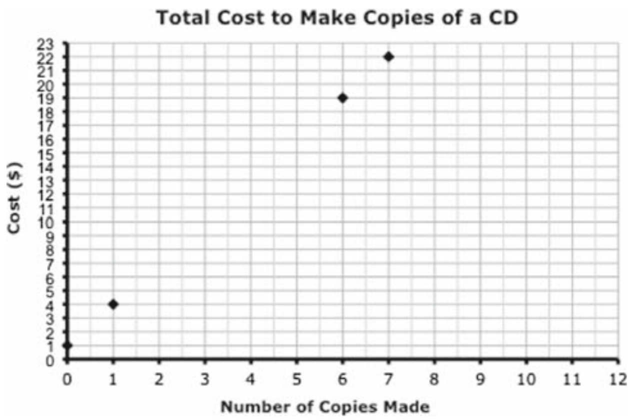
One of the challenges of learning and using mathematical representations is grasping their meaning by recognizing or forging associations between them and other representations, or connecting them to objects and events in the world (e.g., Pape & Tschoshanov, 2001). *Grounding* is commonly used to describe the mapping that a person makes between an unfamiliar or abstract representation, and a more concrete or familiar referent (Glenberg, De Vega & Graesser, 2008; Nathan, 2008). Grounding, as a process, facilitates meaning making for something that would otherwise be taken as abstract and non-denotative (e.g., Glenberg & Robertson, 1999; Harnad, 1990; Lakoff & Nunez, 2000). Grounded representations are desirable because, once imbued with meaning, they depict relations in a succinct way and potentially support the discovery of new relations. Empirical work has shown grounding to be quite robust for enhancing students' understanding of mathematical principles, procedures, and representations, leading to learning gains in areas ranging from basic number and early arithmetic (Griffin, Case & Siegler, 1994), fractions (Saxe, Taylor, McIntosh & Gearhart, 2005), arithmetic and algebra story problem solving (Glenberg, Jaworski, Rischall, & Levin, 2009; Nathan, Kintsch & Young, 1992), and complex adaptive systems (Goldstone & Son, 2005).

While grounding mathematical representations is a highly desirable aim of instruction, there also appear to be downsides. De Bock and colleagues (2003) showed that authentic contexts and self-generated representations—both highly grounded representations for learners—had a negative effect on students' performance in high school geometry problems. Concrete exemplars of a concept also convey superficial information that may distract learners from the underlying mathematical structure, helping with initial learning, but limiting its subsequent transfer (Kaminski, Sloutsky & Heckler, 2008) unless concreteness was reduced over time in favor of greater abstraction (Goldstone & Son, 2005).

Negative effects of grounding have also been shown specifically with Cartesian graphs. When graphs are seen as grounded representations for the student, their shape can be interpreted as depicting the physical movement of the phenomenon being represented rather than as covariation among measured quantities (Clement, 1985; Smith, diSessa, & Roschelle, 1993). Some of the negative effects of grounding to Cartesian graphs, particularly for novices, may be attributed to their nature as an external representation (Zhang & Norman, 1994) where "the information … can be picked up, analyzed, and processed by perceptual systems alone." (Zhang, 1997, p. 180). Thus, it appears that one's perceptions and reasoning processes can be inappropriately *bound* to the representations they are grounded to, which can impose superfluous or incorrect constraints on the representations themselves and the strategies that draw upon them, thereby negatively affecting problem-solving performance or transfer.

1.3 Prior work on reasoning with Cartesian graphs

The Cartesian graph is the rectangular coordinate system depicted by all of the points specified in a plane by the orthogonal relations of the x- and y-*axes* (see Fig. 1 for an example, where the x-axis shows the Number of Copies Made, and the y-axis shows the Cost in Dollars). A review of children's experiences with these graphs shows many areas of confusion, including an over-reliance on a literal or iconic interpretation (Clement, 1985; Smith, et al.. 1993), correlating height and slope (Leinhardt, Zaslavsky, & Stein, 1990), and arbitrarily switching between independent (x) and dependent (y) variables (Beichner, 1994). An anthropological perspective (Roth et al., 1999) shows



a) Describe this picture.
b) Describe how the points in the picture relate to one another.
c) Find the cost to make 6 copies of a CD.
d) Find the cost to make 10 copies of a CD.
e) Find the cost to make 31 copies of a CD.

Fig. 1 Task used in the interview protocol

that Cartesian graphs are not readily interpretable by novices if they are not enculturated into graphing practices.

Graphing practices in school generally fall into one of three dominant types. In the primary grades, instructional emphasis is placed on "reading data" directly from a graph (Curcio, 1987). Subsequent tasks direct students to "read between the data" and draw inferences. In later grades, the emphasis is put on "reading beyond the data" (Curcio, 1987) and abstraction or modeling, where quantitative patterns in the data are captured by concise equations or rules (Bertin, 1983, in Friel, Curcio, and Bright, 2001). Nathan and Kim (2007) recently showed empirical support for this developmental sequence.

One aspect of graphs that may distinguish them from other representations is the salience of their spatial and visual properties. Pinker's (1990) model provides a "bottom up" account of how people's conceptual reasoning about the quantitative patterns represented by graphs can be bounded by its visual qualities. The tight relation between these perceptual and the conceptual processes is consistent with work by others who have examined graphical reasoning from the perspective of embodied cognition (e.g., Gerofsky, 2007; Lakoff & Nunez, 2000; Tversky, 2001). From a sociocultural perspective, Radford (2009) also shows the influences the graph itself has on students' thinking and interpretations. Radford argues that students' sense making of Cartesian graphs entails objectification of the mathematical relationships depicted by culturally and historically constituted signs with particular forms of thinking. Gesture and coordinated speech, along with actions, tone and other modalities, captures the students' interpretation through an integrative iconic exhibition.

2 Research question

The influences of the perceptual and spatial qualities of graphical representations on students' mathematical reasoning and actions are central to this investigation. We explore how students' problem-solving strategies, as revealed by their gestures, speech and writing, help us understand students' perceptions and interpretations of graphs. In framing this question, we explore the ways that representations both foster meaning making (i.e., ground) and inhibit (i.e., bound) reasoning. Our work was motivated by a particular desire to understand how students use Cartesian graphs when reasoning about pattern generalization tasks. Thus, in this paper, we investigate in-depth how middle school students using graphical representations talk about and perform on a set of pattern generalization tasks. Specifically, our inquiry is guided by the following research question:

> How do students' levels of representational fluency and disfluency as revealed in speech and gesture affect their problem-solving processes when generalizing and making predictions from patterns of data presented in Cartesian graphs?

In the analyses that follow, we relate students' problem-solving processes with our coded accounts of the students' pattern generalization strategies as manifest in their speech and gestures. This analysis leads us to recognize that co-occurrences of the uses of certain strategies signal particular representational fluencies and disfluencies that have theoretical importance for distinguishing among students, and that these fluencies and disfluencies imply very different interpretations of graphs that call for different pedagogical responses. We present three student cases to illustrate the different forms of representational fluency and disfluency that we believe to be representative of the varieties of student thinking with graphs present in our data.

3 Methods

3.1 Participants, materials, and procedures

Thirty-eight US students in sixth ($n = 13$), seventh ($n = 15$), and eighth ($n = 10$) grades participated in one-on-one, videotaped interviews with a researcher. Thirteen of the 38 students were from a large, urban middle school with a high percentage of non-Caucasian students (91.7%) and students who were eligible for the national free/reduced lunch program (86%). The other 25 students were from a mid-size, urban middle school, also with a high percentage of non-Caucasian students (86%) and students who were eligible for free/reduced lunch (79.1%). All students were enrolled in at least grade-appropriate mathematics classes, with both of the schools utilizing the NCTM Standards-based *Connected Mathematics Project* curriculum (Lappan et al., 1998) at each grade level. In this study, we focus on students' responses to a task that assessed their abilities to extrapolate from a linear pattern presented on a Cartesian coordinate graph.

As shown in Fig. 1, parts (a) and (b) of the task provided a baseline assessment of students' abilities to identify a Cartesian graph in a general way and interpret the meaning of the information presented. Part (c) assessed students' basic representational fluency with reading a point on a graph. Both parts (d) and (e) were *far prediction* (FP) tasks, because they asked students to come up with a value that exceeded the pattern as it was graphically presented. The answer for part (d) corresponded to a y-value of 31, a value beyond the numerical limit of the y-axis as drawn. Both coordinate values for part (e) ($x = 31$ and $y = 94$) were beyond the numerical limits of the x- and y-axes as drawn.

Students were interviewed individually in the students' regular school during the school day. Before viewing the items, students were instructed to "think aloud" throughout the interview—to say as much as they could about their thinking as they solved the problems. When students were quiet for long periods (greater than several seconds), they were reminded by the interviewer to "think aloud." If a student's response consisted solely of an answer, the interviewer asked the student to explain how he or she arrived at the answer. Students were allowed to use the ruler and calculator provided, but were asked to describe aloud their measurements and calculations.

3.2 Coding

Transana, a software package for analyzing video data (Woods and Fassnacht, 2007), was used to create and code the video transcripts. Coding was completed over multiple passes through the transcripts (cf. McNeill and Duncan, N.D.); first, by segmenting the transcript into short utterances determined by changes in speaker or linguistic cues, indicating change in thought or purpose, and by annotating pauses, breaths and any other non-speech sounds; second, by signifying each gesture; third, by annotating each gesture; fourth, by coding the speech and gestures analytically (see below), rather than descriptively; and finally, by revising codes as the data revealed more information about the speech and gesture production.

For part (a), which asked to describe the picture shown, student responses were coded as: (1) Graph, for responses indicating that the picture showed a graph; (2) Features, for responses that read the graph's title and/or axis labels as description of the picture; and (3) Interpretations, for responses that interpreted the information presented in the graph as a description, such as "The picture shows that you can make 6 copies of a CD for $19." Part (b) responses were coded into categories based on the type of relationship between the plotted points, such as: (1) covariate, for responses that described a dependent relationship between the x- and y-values; (2) x-axis or y-axis, for responses which described only the change occurring in the x- or y-values, respectively; and (3) no relationship, for responses that explicitly indicated there was no relationship between the plotted points. For part (c), students' responses were coded as correct or incorrect to provide baseline information on fluency in reading Cartesian graphs. Because the given x-value and its corresponding y-value were both visible, most students answered this item using direct inspection.

For parts (d) and (e), the far prediction tasks, students' responses were scored as correct/incorrect, and coded for their solution strategies (below), while gestures were coded for boundedness (see below). While all five parts of the interview reveal aspects of students' reasoning with graphs, the analyses presented below focus primarily on parts (d) and (e) because these were the only tasks that invited extensions beyond the patterns as presented, and gave us a look at the nature and influences of students' bounded and unbounded views.

3.2.1 *Strategy codes*

The set of strategy codes used in this study were produced using both top–down and bottom–up methods, and so do not represent all of the possible correct and incorrect strategies that could be employed to solve the tasks in parts (d) and (e). The complete list, along with examples, the relation of each to our notion of representational fluency, and their frequencies of occurrence in the data can be found in Appendix A. The Arithmetic code, with subcodes Recursive Arithmetic, Linear (the expected response to the question), and Linear Combination, were obtained from a review of the literature on strategies used to solve algebraic problems (e.g., Lannin, Barker & Townsend, 2006, Nathan & Kim, 2007). Additional codes, namely Pragmatic, Spatial Estimation, Graph Enlargement, were developed using the *constant comparative* method (Glaser, 1978) of analyzing the speech and gestures in the transcripts.

3.2.2 *Codes for boundedness*

Students' responses on the FP tasks revealed two distinct gesture patterns. Students sometimes used their hands to indicate that their interpretations of the graph was inherently limited by its physical and numerical boundaries; that is, to the student, the information provided within the graph was limited to what was displayed within the grid, bounded by the least and greatest *x*- and *y*-values shown. Gestures that remained within the physical boundaries of the given Cartesian grid were coded as In Bounds or On Bounds, and used as potential evidence of a *bounded* view. In contrast, an alternative code was assigned when gestures did not indicate any physical or numerical restriction on the data presented in the Cartesian graph. These gestures may have referred to values that were beyond the numerical limits of the *x*- and *y*-axes, and exhibited movements that extended beyond the physical limits of the gridlines. Any gesturing that students produced while either solving or explaining their solution to the FP task that referred explicitly to space beyond the physical limits of the Cartesian graph (i.e., Outside Bounds) was considered as evidence that a student had an *unbounded* view of the graph.

If students supported their verbal explanations or their work with gestures that were both bounded and unbounded in nature, the overall classification of their gestured responses was coded as unbounded. It was also the case that a student's gestures may have been coded as bounded even though they may have an unbounded view; many studies in other areas have shown that children indicate one view with gestures and a different one with words (e.g., Alibali & Goldin-Meadow, 1993; Church & Goldin-Meadow, 1986; Roth, 2001). We accounted for this possibility when constructing our categorizations of grounding to the graph presented in Table 2.

3.2.3 Categorizations of grounding to the graph

Analysis of the transcribed and gesture-annotated data aimed to establish whether students indicated that the pattern provided in the graph was bounded within the graph's axes, and whether there was a pattern in the strategies students used to solve both the simpler and more complex FP tasks. From these general patterns of strategy use and gesture production, we sought to understand more about the students who exhibited two kinds of representational disfluencies. One form of disfluency occurred when students who were bounded to a given representation did not alter or translate the given representation. We label students demonstrating this disfluency as *physically grounded,* because they used the graphs and the information presented within them in ways that suggested the graph was fundamentally limited and immutable, as though it were an invariant physical object that can only be read from and referred to. A second form of disfluency was observed when students who showed evidence of a bounded view as determined by their gestures, physically altered the graph using rescaling and extending, but did not translate the information presented in the graph to any type of generalized symbol structure such as the number line, a rule, or an equation. Students demonstrating the second kind of representational disfluency employed a form of unbounded gesturing when adapting the graph, but their gestures subsequently remained within the spatial bounds of the newly altered graph—thus exhibiting a hybrid form of bounded and unbounded gesturing, which we refer to as *spatially grounded*.

A third group of students demonstrated a different kind of fluency with the graphical representation that supported unbounded reasoning. They saw the graph as a portrayal of some pattern that could be reinterpreted through translation to generalized numerical structures such as arithmetic or algebraic formalisms, but that still maintained its connection back to the original graphical pattern. We refer to these students as *interpretatively grounded*. Consequently, we argue that when considering both students' gestures and problem solving strategies, the ways in which students use Cartesian graphs to ground their understanding of the data presented in them fall into three categories of classification, namely physically grounded, spatially grounded, and interpretatively grounded (see Table 1).

Table 1 Criteria used to categorize responses as indicative of one of three types of grounding

Construct	Criteria
Physically grounded	Speech and gesture bounded by physical and numerical bounds of graph.
	No shift to numerical relationship during FP tasks (representational disfluency).
Spatially grounded	Expresses in speech or gesture need to enlarge the graph to solve FP tasks.
	May or may not shift to numerical relationship during FP tasks (representational disfluency).
Interpretatively grounded	Speech or gesture unbounded by physical and numerical bounds of graph.
	Shifts to numerical relationship before or during FP tasks (representational fluency).

4 Results

We present descriptive statistics, as well as exemplary cases, to illustrate the *physically, spatially*, and *interpretatively grounded* ways in which students use Cartesian graphs (see Table 1). In each interview, students responded to a sequence of questions scaffolded in ascending order of complexity, starting with more procedural, skill-assessing tasks and then moving to more conceptual, problem-solving tasks (see Fig. 1). The analyses of the interview data presented here focus on the two FP tasks that necessitated students' reasoning beyond what were explicitly presented in the graphs [parts (d) and (e)]. Friel and colleagues (2001) suggest that procedural tasks ask students to focus their attention on one quantity, while more conceptual tasks require attention to be focused on information across data points. Hence, the conceptual tasks in our study, parts (d) and (e), are also more difficult than more procedural tasks. As the majority of the students in our sample (71%) were quite proficient in correctly solving the procedural task [part (c)] in the interview, the focus of these results is to describe evidence of representational disfluencies when students solve tasks involving pattern generalization.

4.1 The physically grounded case

As described in Table 1, students interpreting the graph in more physically grounded ways used speech and gesture evoking a bounded view of the graph. Of the total responses across both parts (d) and (e), 37.5% ($n = 27$) of the responses used gestures that were bounded within the graph (see Table 2).

To illustrate how a bounded view of Cartesian graphs is evident in students' work, we present selected segments from an interview with an eighth-grade student named Paul. Paul's responses to parts (a) and (b) indicated that he was very familiar with Cartesian graphs, as he stated: "Well I mostly know how to do all this because it's mostly review because this as everyone knows is a graph." Further, he provided a correct response to part (c) without any hesitation, indicating at least a minimal ability to read the graph. From Paul's performance on the first three questions, in particular the third question, he appeared competent in *reading the data* (Curcio, 1987 by demonstrating understanding of the structure and mechanics of Cartesian graphs by successfully completing basic procedures involving their interpretation.

Table 2 Frequency of bounded and unbounded gesture production across FP tasks

$N = 36^a$	Part (d)	Part (e)	Total
Bounded	14	13	27
Unbounded	22	23	45
Total	36	36	72

[a] We had no gesture data for two students for parts (d) and (e)

In part (d), Paul needed to extrapolate from the information given to answer the question "What is the cost to make 10 copies of a CD?" While 10 is within the domain as visually depicted in the x-axis, the cost represented by the point (10,31) lies beyond the frame of the graph. Curcio (1987) refers to this more demanding competency as *reading between the data*.

For part (d), Paul wrote an answer of $23, accompanied by an explanation (see Appendix B). It is evident from Paul's response that he does not attempt to find a numerical relationship between the given points to determine the cost to make 10 copies. His gestures indicate that he is aware of the need to place a point corresponding to 10 copies on an appropriate gridline, complimented by a guess that the point should be at (10, 23) as depicted in lines 4 and 5. From his guess, he wrote an answer of $23 as the cost of 10 copies. Although he refers to his strategy as guessing, his gestures seem to indicate that he knows that the point should be placed on the line $x = 10$, but the range of the y-axis limits his ability to accurately determine where the point should be placed along the line. The gestures he produces during his explanation of part (d) stay within the physical limits of the graph as it is drawn, even though the task asks him to find values beyond those limits, and his verbal responses indicates no attempt to reason more generally from the points plotted or to extrapolate beyond the information given. Thus, as he seems to pick a point given on $x = 10$, but fails to generalize a pattern, Paul's solution was not assigned a pattern generalization strategy code (as listed in Appendix A).

For his response to part (e), Paul quickly acknowledges to the interviewer that he "has no idea." The interviewer proceeds to scaffold Paul's response by drawing a line connecting the points on the graph, extending beyond the uppermost y-value gridlines, with a ruler. She then asks, "Does that help you figure out what the answer might be?" Paul responds, "No, not really." Paul admits to recognizing that she has just drawn a line, but is not sure how that helps him answer the question. Thus, even with the prompt from the interviewer revealing a linear relationship between the points on the graph, Paul does not see a way to extrapolate the pattern as given or to translate the graphical information into a different representation such as a numerical description of the relationship to answer the far prediction question of part (e).

Paul, like other students who may interpret the graph in a bounded way, makes no attempt to alter the graph or translate the information to another representation that might aid his analysis. Figure 2 shows that students who

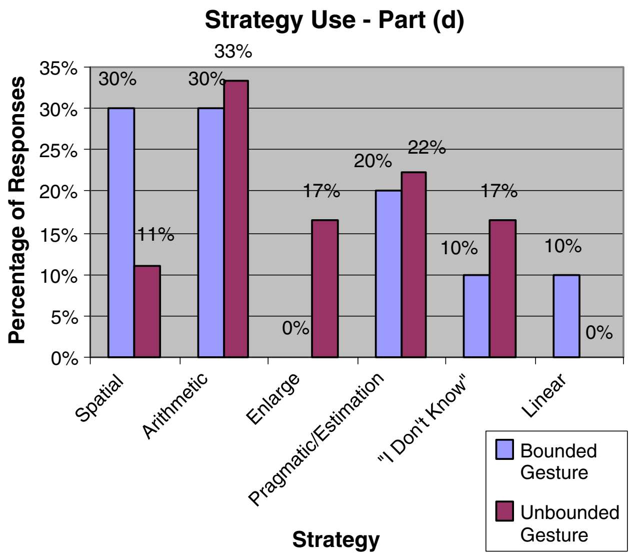


Fig. 2 Comparison between of strategy use and boundedness of gesturing for responses to part (d)

made exclusively bounded gestures when using the graph for part (d) tended to use spatial, arithmetic, and estimation strategies more, a linear strategy less, and an enlarge strategy not at all. Being physically grounded to the graph may constrain students' abilities to apply the mutable features of the graph, such as the scale of the axes, to help them make modifications that can reduce the complexity of the task or translate information from the graph.

4.2 The spatially grounded case

Students who are physically grounded to the graph, like Paul, experienced difficulty in extrapolating patterns from the graph. However, a greater number of the participants ($n = 23$; 64%) were not physically grounded to the graph. Of these students, some still exhibited some representational disfluency, employing strategies of spatial estimation or altering the graph to help them find solutions to far prediction tasks instead of translating the information in the graph to a numerical or symbolic form.

Students' responses were coded as using the Spatial Estimation or Enlarge strategies (see Appendix A) if they either visually estimated where points would be plotted or altered the given representation to meet particular needs, such as a greater range for the y-axis, but did not translate to a new representation. Our data shows that neither strategy was particularly effective for the participants; spatial estimation was used to obtain correct answers on either (d) or (e) only 25% of the time, but enlargement, while a creative solution method that involved physically altering the graph, was never applied successfully. Further, the Enlarge strategy was not frequently employed by the participants; six responses in part (d), and six responses in part (e) were coded as Enlarge where only two students used the Enlarge strategy for both responses. Thus, to get a better sense of students' performance, the ways in which students switched strategies between parts (d) and (e) more accurately reflects how students employed particular strategies. Table 3 shows strategy switching by students from part (d) to (e).

In Table 3, the cells that are shaded gray indicate the number of responses coded as the same strategy across both FP tasks. For example, four students who answered part (d) using an Enlarge strategy ($n = 6$) also used the Enlarge strategy to answer part (e). Only one of these six students ultimately translated to a Linear strategy to answer part (e). Also, students were more likely to switch strategies if having answered part (d) with either a Linear or Pragmatic strategy (67%), than for students who answered

Table 3 Strategy "switching" (or maintenance, in gray fields) across parts (d) and (e)

Part (e) → Part (d) ↓	No response	Pragmatic	Spatial Estimation	Enlarge	Arithmetic	Linear	Total
No response	2	0	1	0	0	0	3
Pragmatic	0	3	1	0	4	1	9
Spatial Estimation	1	0	4	1	1	0	7
Enlarge	1	0	0	4	0	1	6
Arithmetic	1	0	0	0	9	1	11
Linear	1	0	0	1	0	1	3
Total	6	3	6	6	14	4	39

All entries represent number of student responses part (d) with Spatial estimation (43%), Enlarge (33%) or Arithmetic (18%) strategies.

To further elucidate the construct of spatially grounded to the graph, we present a case featuring Sophia, a sixth-grader attending the same school as Paul. Like Paul, Sophia provided a correct answer to the point interpretation task in part (c), hesitating only 5-s after the interviewer had posed the question. The FP tasks in parts (d) and (e) posed more of a challenge. Unlike her response to part (c), Sophia exhibited some apprehension and uncertainty when responding to part (d) (see Appendix B).

Sophia began her response to part (d) by pausing a notable 11 s after the interviewer stated the question. It is clear that Sophia realizes that the correct answer is greater than $23.00, but she is unable to use the graph to—literally—*find* the answer. Throughout her verbal report, her gestures remain within the boundaries of the grid formed by the *x*- and *y-axes*. In Sophia's case, *finding* the answer corresponds to seeing the answer spatially on the graph. This may explain why she indicates a desire to extend the *y*-axis in lines 13–17.

Sophia's response to part (d), a Graph Enlargement strategy, shows a spatially grounded use of the graph as we have defined this construct. There was indication throughout her responses to the interview questions that she could read points off the graph with facility, but there was no indication that she had developed a mathematical rule linking the points placed on the graph. While her verbal response was definitely beyond the numerical limit of the *y*-axis, the bounded nature of her gestures indicated a persistent need to be spatially grounded to the graph. Further, Sophia's need to extend the graph to answer far prediction questions persists in her response to part (e) in Appendix B. Although she declines to provide an answer to the fifth question, she again describes a method of extending the graph that could help her to find a solution coded as the Enlargement strategy (see Appendix A).

Unlike Sophia, the majority of students employing an Enlargement strategy actually performed the enlargement, usually by drawing extra columns to the right of the maximum point shown along the *x*-axis and adding extra rows beyond the maximum point shown along the top of the *y*-axis. Another interesting aspect of the interview is her use of the word "it" in line 9. One interpretation of what she might be referring to as "it" would be the spatial pattern between the dots plotted.[1] Nevertheless, this statement also indicates that she is at the mercy of what she sees on the graph, rather than in control of manipulating the information she extracts from the graph.

4.3 The interpretatively grounded case

While spatially grounded students may be able to answer FP tasks using Cartesian graphs to the extent that the physical affordances of the size of the paper, or perhaps the program of a graphing technology, will allow, it is necessary for students to be able to generalize to more abstract, symbolic representations to fully realize the power of mathematics. As stated in the 2000 National Council for Teachers of Mathematics *Principles and Standards for School Mathematics:* "One of the powerful aspects of mathematics is its use of abstraction—the stripping away by symbolization of some features of a problem that are not necessary for analysis, allowing the 'naked symbols' to be operated on easily" (p. 69). Thus, as students progress in their understanding of relationships between graphs and symbolic equations, they will be more *interpretatively grounded* to the graph. Students who exhibit this type of grounding translate the graph into a more mathematically abstract representation, such as an equation, and connect the equation to the graph to make sense of the outputs that result when manipulating data in the equation. In other words, interpretative grounding capitalizes on the affordances of representational fluency.

The most effective strategies were Linear (which led to a correct answer 50% of the time) and Arithmetic (which had a 35% success rate). Both strategies required students to translate the quantitative information from the graphical representation to a numeric or algebraic representation, thereby exhibiting one form of representational fluency. For part (d), as Fig. 2 shows, students who were coded as giving unbounded gestures never used the most effective and most general method, the Linear strategy. These students tended to use arithmetic methods (33% of the time), estimation methods (22%), and the enlargement strategy (17%). Unbounded codes were also more closely associated with "I Don't Know" responses (17%), indicating that, while these students imagined the pattern continuing beyond the graph as depicted, they did not have the analytic skills to determine how to proceed.

We found that part (e) was a more complex task than part (d). Over half (53%) of the 38 students in our study correctly solved the part (d) item, while only 26% were successful with part (e). While boundedness in gesturing did not seem to play a role in students' strategy choice for the simpler FP task [part (d)], a similar analysis of the more complex FP task (Fig. 3) showed that those with bounded gestures now used the Linear strategy in lower numbers

[1] Another possible interpretation of Sophia's use of the word "it" in line 9 is that she is referring to the graph as a whole unit, and views either the graph or the points plotted as malleable and organic.

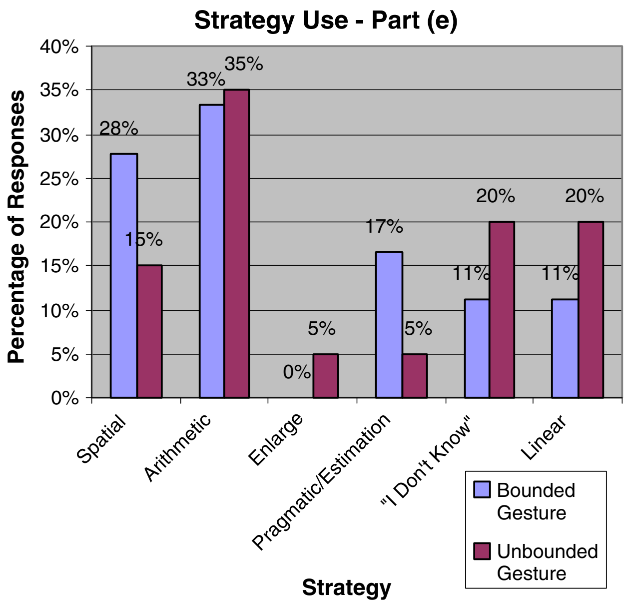


Fig. 3 Comparison between of strategy use and boundedness of gesturing for responses to part (e)

than those responding with unbounded gestures in reference to the graph. On one hand, we might expect the more difficult FP tasks to lead to greater use of more sophisticated, generalized ways of working with the data presented—such as a Linear strategy. On the other hand, it is worth noting that a noticeable change only occurred for the group with unbounded gesturing, suggesting a link between the production of unbounded gesturing and greater representational fluency. That this relationship is reliable is shown by a significant chi-square test comparing bounded and unbounded gesture use with strategy changes, $X^2(1) = 12.6$, $P < 0.001$.

In the following case, the responses of 1 seventh-grade student, Isabel, to the same set of tasks completed by Paul and Sophia, fit our criteria of what constitutes interpretative grounding to the graph. In Isabel's responses, she appears to shift to a more mathematically abstract representation of the data in the graph during the interview. However, her translation of the data is flawed. Thus, while being interpretatively grounded to the graph is the desired behavior, it is not sufficient for correct performance on FP tasks.

The differences between Isabel's ability to flexibly work between representations and the challenges that Paul and Sophia faced to do the same are evident in Isabel's response to the less complex FP task in part (d). In her response (see Appendix B), we see how Isabel uses generalizations she has already made about the graph, and questions these generalizations as she translates from the graph to a numerical representation. In lines 6–7 of her response to part (d), Isabel questions her strategy of multiplying the number of copies by three, by identifying information in the graph that does not match the pattern she has established. In particular, she recognizes that 7 times 3 is 21, not 22 as shown in the graph when $x = 7$. Until this moment, Isabel had not identified the role of the y-intercept in establishing the linear relationship. She likely had just been using the differences in the y-values of the points plotted to determine the linear relationship—which happened to be the value of the slope of the line because the x-values of subsequently plotted points differed by 1. She used this difference as a factor to multiply by y-values of given points, a form of the strategy coded as Arithmetic (see Appendix A).

What is quite surprising about her response is how quickly she determines the information she overlooked, indicated by pointing to the y-intercept in line 9. She uses the graph as a "check" on her symbol representation, moving between abstract, mathematical computations and the spatial properties of the graph to make sense of results. Her ability to move flexibly between representations is typical of those who are interpretatively grounded to the graph.

At the end of the excerpt, in lines 16–21, Isabel described her new strategy known as the "recursive arithmetic" strategy (e.g., Lannin, Barker & Townsend, 2006) as described in Appendix A. While this method is less sophisticated than creating a linear equation like $y = 3x$ to describe the relationship between the points, it is nonetheless appropriate for answering the question. Isabel's confidence in her solution strategy is evident in her response to part (e), "What is the cost to make 31 copies of the CD?" Without even referring to the graph for information to help her solve the problem, she employs the recursive arithmetic strategy she developed while solving part (d). As her response to part (e) shows (see Appendix B), at this point in the interview she has shifted from relying on the graph to interpolate and extrapolate, to using a more abstract and efficient method.

Much of Isabel's response to this question indicates that, while not possessing a complete understanding of linear relationships, she is comfortable with both graphical and symbolic representations. Her verbal and gesticulated responses in lines 12–15, "it is the same thing because it is still the same graph," show that she understands her mathematical equation is representative of the visual pattern. Following Curcio's (1987) framework, she *reads* from the graph. In its explicit form, her response shows that she believes that the pattern is consistent, regardless of whether she uses the graph or her arithmetic strategy. She clearly indicated the presence of a y-intercept as evident in lines 6–8. Furthermore, she *interprets* the meaning of the y-intercept for this graph in lines 8–9, a clear indication that she is interpretatively grounded to the graph. However, her verbalized strategy in lines 3–4 makes it difficult to claim that she has conceptualized these points as sharing a linear relationship. She stated that the *y*-intercept "messed [her] up" in lines 9–10, instead of recognizing the influence of the *y*-intercept on all points plotted. Isabel seems to have an emerging understanding of linear concepts because she identifies the source of her confusion to be the *y*-intercept before adjusting her calculations by adding 1 to make her answers fit the data.

5 Conclusions and implications

One of the major goals of algebra instruction is to develop students' competencies with and across formal representations (e.g., Nathan & Kim, 2007; National Council for Teachers of Mathematics, 2000; RAND, 2003). Our study serves to illustrate the potential of visual grounding that the Cartesian graph provides, and to shed light upon how students negotiate the affordances and constraints of the Cartesian graph when solving problems, which require extrapolation from the information given in the graph. Grounding provides learners with a way to map the unfamiliar to the familiar, and thereby facilitates meaning making. However, grounding to concrete instances and interpretations can have a negative effect on later problem solving (Goldstone & Son, 2005; Kaminski et al., 2008). Pinker (1990) and Zhang (1997) argued that one's reasoning may be overly influenced by a representation's visual qualities, rather than the conceptual relations that the spatial representation is meant to convey. In the case of the physically grounded student, Paul, we see that the ability to solve FP tasks is inextricably linked with what information the Cartesian graph can visually provide. Hence, the student falls victim to a key constraint of the Cartesian graph: a finite "frame" or "window" for depicting information. For Sophia, the spatially grounded student, the "frame" of the Cartesian graph is mutable. The graph's axes can be extended, and if the axes and labeling remain consistent, the extended region can be used to *see* solutions to formerly FP-like tasks. For spatially grounded students, the constraint of having a finite "frame" can be manipulated with the assistance of technology such as the graphing calculator to the point where it becomes an affordance of Cartesian graphs, namely that there is no theoretical limit to the viewing window. However, the most sophisticated, and efficient, use of Cartesian graphs is demonstrated by students like Isabel who are interpretatively grounded to the graph. Students like Isabel deal with the physical constraints of the graph by translating the information provided into representational forms that are not visually bounded, such as algebraic equations, which can support patterns of unlimited domains and ranges. For these students, graphs are used to interpret patterns in the data, not as a primary means for extrapolating from the data. In this sense, graphs truly serve in the generalization of numerical patterns.

The cases we present suggest many implications for early algebra instruction. First, while grounding is often the objective of mathematical instruction, there are negative effects that should not be overlooked. Second, our data show that gestures along with speech provide rich insights into how people encode and interpret representations. Specifically, gesturing limited to the physical bounds of the graph may indicate that the student views the graph as inherently limited by its physical form, and may serve as a signal that instruction should emphasize the theoretically unbounded nature of the functions portrayed in the graphs.

Our study certainly suggests that building competencies in using Cartesian graphs as a tool for generalizing patterns requires continual and consistent pedagogical effort. We speculate that one way to assist students in recognizing the limitations of Cartesian graphs and developing skill in graphical pattern generalization is to include more opportunities for students to encounter far prediction tasks early and frequently in their math education. While Cartesian graphs are a useful tool for charting data and manipulating geometrical forms, they are not the only tool for finding relationships among data. Mathematical work that focuses solely on having students graph equations of lines and/or plot data points without eliciting an equation or generalized rule to represent the relationship among the data plotted may misrepresent the value and purpose of creating Cartesian graphs.

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Appendix A: Students' pattern generalization strategy codes

See Table 4.

Table 4

Strategy name	Description	Example response	Relationship to representational fluency	Frequency [both parts (d) and (e)]
Arithmetic[a]	A constant rate of change between points, with no reference to influence of *y*-intercept	"The points go up by 3 every time"	Translate to generalized numerical pattern	$n = 14$
Linear combination[a]	Decomposes target *x*-value into smaller amounts. Determines *y*-values for smaller *x*-values, then combines those *y*-values to determine *y*-value for target *x*-value	"Since I need to find the cost to make 31 copies, I took the cost to make 10 copies, which was $30 then I multiplied it by 3 to get $90, then I added 1 so the answer is $91"	Fixed to graphical representation	$n = 6$
Recursive arithmetic[a]	Determines the constant rate of change between subsequent points plotted and, from the *y*-value of a point, adds on the constant rate of change until the target *x*-value has been reached	"So if $y = 19$ when $x = 6$, and the points differ by 3, then 19, 21, 24, 27, 30, *y* must by 30 when $x = 10$"	Fixed to graphical representation	$n = 7$
Linear[a]	Finds the cost to make 0 copies and then finds the constant rate of change in cost as the number of copies increases by 1	"I know for every copy you make the cost goes up by $3, and the cost to make no copies is $1, so 31 copies would be 3 times 31 which is $93 plus the initial cost which is $1 so $94 dollars	Translate to generalized numerical pattern	$n = 8$
Pragmatic	Grounds the information in the graph to real-world experience; uses "common sense" to determine solutions	"So, 1 CD is like 5 bucks, so it would cost like 150 or 200 bucks to get it done, to make 31 CDs"	Translate to alternative context	$n = 9$
Spatial estimation	Uses visual patterning of point placement to estimate where the target point would be placed	"So it looks like the points fall along this line. [places ruler flush with points] If I follow this line up to where $x = 10$ crosses it, it looks like the point would be placed around $y = 30$"	Alters graphical representation	$n = 16$
Graph enlargement	States that the solution can be found by adding the necessary amount of rows and/or columns to plot the target points	"To find 31 copies, I would have to draw out the *x*-axis to 31 and increase the *y*-axis to see what it would do"	Alters graphical representation	$n = 12$
Rescale	Adjusts scale of *x*- and/or *y*-axis to obtain greater minimum and maximum values	Crosses out values for 1, 2, 3, 4, 5, 6 on *x*-axis and replaces these values with 13, 14, 15, 16, 17, 18, respectively	Alters graphical representation	$n = 1$

[a] Codes that were obtained from prior research on algebraic problem solving

Appendix B: Transcripts of student responses[2]

Paul's response to part (d)

Oh I don't know. I just **guessed** (places eraser tip on lower portion of gridline for x = 10) since, you know, **6 CD's** (taps x = 6 with eraser) you **pick the** (points eraser tip up to dot for (6,19)) dot right there…19 bucks. Might as well just….**right there** (taps x = 10 with eraser tip twice, then points up the gridline for x = 10 to the uppermost point on the graph (10,23).

Sophia's response to part (d)

Sophia: (Long pause) Like, I like did, like, an **estimate** (thumb and middle finger span y=18 to y=23 on left side of *y*-axis) question. Like a guess. And I **guessed** (touches x=10 on *x*-axis with second finger of left hand) if like you had 10 copies and 6 copies is $19.00 and like **24** (thumb and middle finger of left hand span y=16 to y=23 on left side of *y*-axis), if you **make** (touches x=10 on *x*-axis with second finger of left hand) 10 copies, has to be (touches top of *y*-axis with left hand) like $23.00 or **more** (spans thumb and middle finger alongside left side of *y*-axis with thumb at (0,23) and middle finger slightly higher above the axis. Flicks middle finger vertically while saying "more". Then returns thumb and middle finger to original span).

Interviewer: OK. And how do you know that?

Sophia: Because, because **it** (touches second and middle fingers of left hand at the bottom of *y*-axis) like has to **increase** (moves left hand from area around (0,1) and (1,4) and diagonally to area for (6,19) and (7,22); taps above graph, above the word 'Copies' with middle finger) more."

Sophia's response to part (e)

Interviewer: So now I'm going to ask you the bigger number. Thirty-one copies of the CD, do you know how much it would cost?

Sophia: No, not really.

I: No. Can you say how you might figure it out?

Sophia: Like, you could **like** (touches left hand to label for *x*-axis) expand the graph **to** (slides from left to right on *x*-axis) 31 copies and be **like** (spans left hand along top and bottom of *y*-axis), like **23, 24** (touches label for y = 23 with finger while saying "23"; slides finger up slightly while saying "24") you know, **like to** (slides middle finger vertically a bit) see what it would **do** (taps top of *y*-axis with middle finger). To 31 copies.

[2] Gesture events coincide with boldface text and are described in parenthetical statements.

Isabel's response to part (d)

Interviewer: Can you find the cost to make 10 copies of a CD?

Isabel: {No speech - writing response} (00:29)

I: OK, and can you tell me how you did that?

Isabel: Well since it goes up **by 3** (motions with top of pen in right hand toward graph) every time and **since** (motions with top of pen in right hand toward graph) 8 times, wait, OK. Wait. That doesn't make sense. That's not right.

I: Can you share what you're thinking?

Isabel: I'm thinking that that's not right because 7 times 3 does not, isn't 22. It'd be 21 so because of **that right there** (touches *y*-intercept with second finger of left hand) it, OK, so 7, it'd be $31.00. {**Whispering**}(counts by placing thumb, then index finger, then middle finger of left hand along left side of response sheet as she writes with right hand after each counting triad; repeats counting procedure 4 times)

I: Can you explain how you figured that out?

Isabel: I just added 3 **onto each one** (moves uncapped pen point with right hand in three tapping moves along sequence of three intermediate results on written response) from the previous one. Since **that was 22** (touches pen cap end to '7=22' on response) I added 3 which **made it** (touches pen cap end to '8=25' on response) that 25, and 25 plus 3 is 28 and then 28 plus 3 is 31. And that's how I got the answer.

Isabel's response to part (e)

Interviewer: So now I'd like to ask you can you find the cost to make 31 copies of a CD.

Isabel: Now I think I can I've got this one, 31 times 3 and then I'll just add one at the end. There.

I: So just explain to me how you did it.

Isabel: Well, since last time it wasn't 30, but it was 31, **since that** (touches bottom end of pen in right hand to bottom of *y*- axis and moves pen back and forth twice from bottom of *y*-axis to (0,1)) it, just to make no copies… **it, it's** (taps second finger of left hand on (0,1)) a dollar. I decided **since it messed me up** (moves bottom end of pen in right hand above the graph along the middle, hovering over dot for (6,19)) last time thinking it was 30 and I found out at the end that it was 31, I decided that, **that** (touches response '93+1=94' on paper) **is the same thing** (moves bottom of pen toward graph over area where dots are located) because it still is **the same** (circles pen above graph in large circles encompassing most of the graph 7 times) graph. So I added one after I timesed it.

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