GROUNDED MATHEMATICAL REASONING

Mitchell J. Nathan

University of Wisconsin–Madison

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Introduction

Galileo said, "[The Universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word." (*Opere Il Saggiatore* 1618/1844: 171). How then, are we to comprehend mathematics itself? Grounded cognition provides an account.

Grounded cognition, notes Barsalou (2008: 619), explores the assumption that intellectual behavior "is typically grounded in multiple ways, including simulations, situated action, and, on occasion, bodily states." When body states and body-based resources are central to these accounts of intellectual behavior, scholars typically use the more restricted term, *embodied cognition*. Grounded cognition, the more general term, is often contrasted with models of cognition that are based primarily on processes and structures composed of symbol systems that are abstract, amodal, and arbitrary (also called "AAA symbols;" Glenberg et al. 2004), because the symbols are non-concrete, make no reference to physical or sensory modalities, and serve their representational roles through arbitrary mappings between the symbols and the phenomena to which they refer. Examples of models based on AAA symbols include feature lists, semantic networks, production systems, and frames (Barsalou & Hale 1993).

I. VARIETIES OF MATHEMATICAL REASONING

By its very nature, mathematical reasoning (MR) addresses behaviors that arise from one's imagination by entertaining possible actions on imaginary (i.e., symbolic) entities (Nemirovsky & Ferrara 2009). MR entails reasoning with and about abstract symbols and inscriptions that denote these imagined entities and their interrelationships. While MR may seem monolithic, it includes a vast space of behaviors, from very primitive processes involving approximate number and small-number cardinality that are exhibited by a variety of animal species and human infants, to collaborative discourse and debates over abstract conjectures about universal truths regarding imaginary entities distributed across a range of representational systems and computational resources, that involve multiple disciplinary communities, which may ensue over years and even centuries. Attending to the varieties of MR is important for evaluating far-reaching claims about its grounded nature. In some cases, the influences of the body and body-based resources such as spatial and temporal metaphor on MR seem to be clear and essential. The use of concrete manipulatives and gestures are examples. In highly abstract, and contemplative forms of MR, such as generating and critiquing a mathematical proof, the role of the body and body-based action seems less certain. Thus, one goal of the framework proposed here is to consider how broadly or narrowly grounding influences the nature of MR.

At a minimum, consideration of the grounded nature of MR entails three relevant dimensions. First, is content: Areas such as numbers and operations, arithmetic, algebra, and geometry, which populate the discipline of mathematics (CCSSI 2013; Math NSDL 2002). Second, is the set of disciplinary practices that characterize doing mathematics, such as precisely executing procedures, finding patterns, using strategies and representations (CCSSI 2013; Kilpatrick et al. 2001; NCTM 2000). Third, reasoning about specific content and employing particular practices engage psychological processes. Some processes, like attention and working memory, are generally applicable to all MR, while others, like subitizing, procedural fluency, and comprehension, seem specific to particular content areas and practices (see Table 1).

Table 1. Summary of the MR review. Solid bullets are for general MR content. Open bullets are for sub-topics.

Content	**Practices**	**Grounding System**			**Psychological Processes**	
		Body State & Actions	**Spatial Systems**	**Language & Social Interactions**	**Time Scale**	**Level**
Number		•	•	•		
Approximate Number	Quantification	◦	◦		Milliseconds	Biological
Exact Number	Precision			◦	Seconds	Cognitive
Counting	One-to-one Correspondence	•	•	•	Seconds	Cognitive
Arithmetic		•	•	•		
Small Number	Procedural Fluency	◦	◦		Seconds	Cognitive
Large Number	Strategies, Structure			◦	Minutes-Hours	Rational
Rational Number	Connections, Sense Making			◦	Minutes-Hours	Rational
Algebra		•	•	•		
Equations	Strategies, Representation	◦	◦	◦	Seconds	Cognitive
Story Problems	Problem solving, Comprehension	◦		◦	Minutes-Hours	Rational
Geometry Proof		•	•	•		
Ascertaining	Insight	◦	◦		Milliseconds	Biological
Persuading	Argumentation, Communication		◦	◦	Days-Months	Socio-Cultural

The psychological processes that support the MR content and practices draw across many different systems. For example, investigation of the nature and development of *number sense* (taken up in more detail later in this article) shows evidence for complementary systems: One is sensitive to approximate magnitudes of quantities, develops very early in humans and other species, and is language- and culture-independent; while the other is an exact number system that develops later, serves precise calculations and forms of reasoning, and is mediated by language (Dehaene 1997).

By arranging the dominant psychological processes that support MR along a logarithmic time scale of human behaviors (Nathan & Alibali 2010; Newell 1994) it is possible to cluster MR content and practices with scales of behavior (Table 1). Early developing and rapid forms of MR content, such as small approximate number processing, measurement, counting, and basic numerical operations occupy the biological and cognitive bands, supported as they are by rapid perceptual, biological, and simple computational processes such as subitizing, ANS, and routinized, knowledge-lean problem solving. Interactive, extended, and distributed forms of MR content, such as complex modeling and proof practices, aggregate in the longer time bands dominated by rational and sociocultural scales.

Conceptually, larger scale processes supervene on finer-grained ones. In practice, temporal bands demarcate distinct fields of study, sets of research methods, and discourse communities (Nathan & Alibali 2010). The division between cognitive (seconds) and rational (minutes-hours) bands is particularly notable to this discussion. Behavior in the rational and socio-cultural bands is no longer context free -- no longer determined primarily by the internal workings of the mental system -- as it is at the cognitive and biological bands. Rather, behavior at larger time scales is shaped by constraints provided by context, including specific tasks and goal structures, context-specific knowledge, tool use, cultural norms, and social interactions (Newell 1994).

II. GROUNDING THE FORMS OF MATHEMATICAL REASONING

Lakoff and Núñez (2000) propose an account of the origins of mathematical ideas as grounded in people's sensory and perceptual experiences and body-based actions. Basic mathematical ideas, such as addition-as-collections-of-objects and sets-as-containers, serve as *grounding metaphors* that require little overt instruction and serve as the foundation for more complex ideas. *Conceptual metaphor* is a neural mechanism that enables people to map the conceptual structure of one domain (e.g., arithmetic) to support reasoning about a different domain (e.g., geometry). Abstractions arise when conceptual metaphors from concrete percepts, actions, and experiences are applied, layer upon layer, to support inferences about the behaviors of symbols and other notational inscriptions. Algebraic equations, for example, can be "solved" when algebraic "objects" are "moved" in ways that isolate an unknown quantity while preserving the "balance" of the equation (Nathan 2008).

Number Sense

As noted, infants and many nonhuman primates and other animals exhibit number sense. They can be trained to respond to the cardinality of one set of stimuli -- objects of a certain color and shape, for example – and then respond appropriately when the arrangement of objects, their size, color and shape are dissimilar, as long as cardinality is maintained. These individuals can also match cardinality across modalities, as when investigators match numbers of objects to tones of varying pitch and duration, or flashes of light. Macaques, for example, can identify the proper number of voices for a given number of faces (DaHaene & Brannon 2011). Key to these universal feats is that the set of objects and events must remain small, typically 5 or less.

Humans and other species are sensitive to the magnitudes of numbers, even when those numbers are presented symbolically. In seminal work, Moyer and Landauer (1967) showed that numerical judgment times correlated with number magnitude. When comparing numbers, response times were longer when their relative magnitudes were close together (2 vs. 3 was slower than 2 vs. 9), and with increasing magnitude for the same relative distance (2 vs. 3 was faster than 8 vs. 9). This correlation should not occur if numerals were, strictly speaking, amodal symbols. Observations of this pattern of number judgment behavior across a range of stimuli in humans of western and indigenous cultures, and in other species, has led researchers such as Dahaene & Brannon (2011) to posit the existence of a language- and culture-independent *approximate number system* (ANS) for representing numerical magnitude (Gallistel & Gelman 1992) in an analog manner.

Behavioral data also reveal a Spatial-Numerical Association of Response Code (SNARC) effect, whereby numerical magnitude is analogically represented along a spatial continuum (Dehaene et al. 1993). For example, responses for numerical judgments on smaller magnitudes that do not address magnitude (e.g., parity judgments) are faster for the left hand, and responses for larger ones are faster for the right hand. Brain-imaging studies show that tasks of approximate number arithmetic activate regions typically associated with visuo-spatial processing (DeHaene et al 1999).

For all its power and universality, however, a system for representing and manipulating magnitudes cannot produce the precise, interconnected, and self-referential system that we identify as the discipline of mathematics (Núñez 2009; Rips et al. 2008). It is necessary to perform *exact mathematics* that mediates accurate counting of large numbers, and precise computational procedures. Behavioral and brain-imaging studies show the exact number system is mediated by language (Dehaene et al. 1999; see Figure 1). Patients with language impairments tend to exhibit impaired performance with exact MR, though their reasoning about approximate numbers may be intact. Bilingual (Russian-English) adults trained on either exact or approximate sums in one language or the other showed that access to exact arithmetic information was affected by language format, though this was not the case for approximate number information (DeHaene et al. 1999).

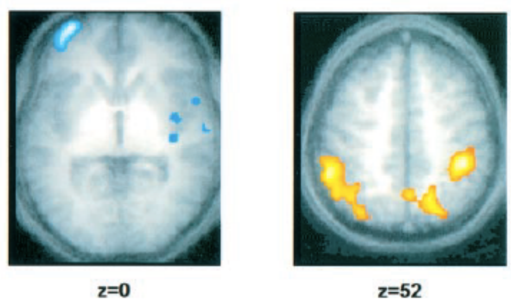

Figure 1. Neuro-imaging (fMRI) data showing dissociation during exact (blue; word-association regions) and approximate (yellow; visuo-spatial regions) calculations. (Image taken from DeHaene et al. 1999).

Scholars (Dehaene 1997; Lemer et al. 2003) suggest the existence of two dissociable systems: An *analog* system for approximate number magnitude; and a *digital* system for exact numerical reasoning and computation. Case and colleagues (e.g., Case et al. 1996) have argued that the integration of number words along with counting activity and magnitude perception are necessary for the conceptual development of number.

Counting

Counting is a skill that develops over time. The agent doing the counting must learn to keep track in order to preserve one-to-one correspondence (Gelman & Gallistel 1978). For children, hand gestures prove to be important in this developmental process (Butterworth 1999; Saxe & Kaplan 1981). Gestures help children to keep track of which items have been counted and which have yet to be counted, and also help coordinate uttering the number word with the appropriate tagging action (Alibali & DiRusso 1999). More broadly, finger discrimination (e.g., digital gnosia) among 5 and 6 year olds is a strong predictor of later math performance (Fayol, Barouilette & Marinthe 1998). Adults apparently retain the symbol-finger association of digital counting behaviors and exhibit muscle activity of the left hand for small numbers (1 through 4) and of the right hand for larger numbers (6 through 9) (Soto et al. 2007).

Counting is grounded in one's language and cultural practices, as well as one's body-based actions. An intriguing case is the Oksapmin of Papua New Guinea, who use 27 body parts as they transcend in regular order from the thumb of one hand along the fingers, arm and upper torso and head down to the pinky of the opposing hand (Saxe 1981). This body-based system is used for counting, numerical comparison, and has been appropriated for monetary calculation as they came in contact with other cultures (Saxe 1982).

Arithmetic with Whole Numbers

Finger movement among children and adults performing mental arithmetic suggests that vestiges of counting behaviors are activated, and possibly mediate, computation (Domahs et al. 2008). Adults responding verbally showed greater difficulty performing mental addition problems when the sum of the unit digits exceeded 5 (e.g., 3 + 4 = 7) as compared to items with no such a break (e.g., 5 + 2 = 7), even though there was no visible use of fingers during the trials (Klein et al. 2011). Behavioral differences at the 5 break boundary are notable since they indicate the role of hands in grounding arithmetic.

Yet language, too, grounds computation. Cross cultural studies show that exact arithmetic calculations among indigenous cultures proceed well when computational demands remain within the range of the linguistic system, but decline when tasks exceed the range of culturally privileged number words (Gordon 2004; Pica et al. 2004). Members of the Pirahã tribe, who use a "one-two-many" system of counting, show degraded performance with calculations above 3. For word problems, performance of both unschooled and schooled children improves when numerical relations map to the appropriate cultural support system (money or time; Baranes et al. 1989; Carraher et al. 1985).

Arithmetic with Rational Numbers

Investigations of early development of arithmetic that point to the mediating role of the body and language systems are typically designed around additive relations. Multiplicative reasoning is also an essential component of MR, and important for the development of concepts of rational number, probability, and algebra.

Although there is empirical support for early developmental reasoning about ratios and proportions through magnitude estimation (McCrink & Wynn 2007), multiplicative reasoning is very demanding for children and adults (Bonato et al. 2007; Smith, Solomon, & Carey 2005). Edwards (2009) showed how gestures of pre-service teachers iconically conveyed the conceptual structure of fractions. Representations of multiplicative relationships that are very realistic (e.g., photographs) can hamper proportional reasoning, while more schematized representations (e.g., diagrams) may invoke more analytic forms of reasoning that support more sophisticated MR (Schwartz & Moore 1998).

Martin & Schwartz (2005) showed that actions using tiles as concrete manipulatives (compared to pictures of the tiles) increased problem solving performances and later learning of fractions for 10 year olds. The advantages of performing actions with manipulatives were not due to cognitive offloading, but because students rearranged tiles in service of interpreting the mathematical relations. These findings led Martin (2009: 143) to conclude that "actions and interpretations develop each other, and eventually, children develop stable ideas and solve problems using mental strategies."

Algebra Expressions and Equations

Algebra equation solving seems to be a quintessential symbol manipulation task (Lewis 1981), where rules are applied in a goal-directed manner until the equation is solved. Algebra distinguishes itself from arithmetic in that symbols or blanks are included to indicate quantitative relations about unknown and changing quantities (e.g., Carpenter, Franke, & Levi 2003; Kaput Carraher, & Blanton 2008; Riley, Greeno, & Heller 1983).

Algebraic expressions and equations simultaneously depict process (*subtract 6 from Sarah's age*) and structure (*always six years younger than Sarah*) (Kieran 1990; Sfard 1991). Children interpret algebraic expressions as processes before they apprehend their structural qualities, which has ramifications for their abilities to solve equations (Knuth et al. 2005; 2006).

Gestures provide important insights into algebraic reasoning. Students' gesture patterns reveal whether they attend to the equal sign of an equation as demarcating the equivalence relation among two expressions (Alibali & Goldin-Meadow 1993). Manipulating students' gestures can alter their interpretations of the equations (Goldin-Meadow et al. 2009).

Interpretations of algebraic equations are influenced by perceptual relations and implied actions (Kirshner & Awtry 2004; Landy & Goldstone 2007). For example, manipulations of a visual background grating moving in a direction compatible or incompatible with the implied movement of symbols for solving an unknown variable affects accuracy (Goldstone, Landy & Son 2010). Animations that provide situated referents of symbolic equations improve performance and transfer on equation solving tasks (Nathan, Kintsch & Young 1992). Thus, despite their formal structure, symbolic expressions invoke perceptual and action-based interpretations that are consequential for MR (Nathan 2012).

Algebra Story Problems

The conventional wisdom for many years was that story problem solving could be divided into processes of story comprehension and equation solving (see Nathan et al. 1992 for a review). Since equation solving was considered a proper subset of story problem solving as a whole, it stood to reason that story problems were necessarily more difficult than solving equations. Two startling findings turn the conventional wisdom for story problem solving on its head. First, performance is *higher* for algebra story problems than for carefully matched algebra equations (Koedinger & Nathan 2004). This comes about because students adopt more reliable solution strategies when presented with story problems, and make fewer arithmetic errors. Linguistic forms are helpful in MR, even when they are devoid of the contextual referents of stories that presumably help with abstract reasoning (e.g., Cheng & Holyoak 1985). Word equations -- items which present quantitative relations verbally without context, such as *Starting with some number, if I multiply it by 6 and then add 66, I get 81.90* -- exhibit a similar performance advantage over (matched) symbolic equations ($6X + 66 = 81.90$) as do story problems. The locus of effect for word equations is similar as for story problems: They invoke more reliable and meaningful solution methods, and tend to avoid arithmetic slips and computational errors.

Geometric Proof

Despite its typical placement in math curricula, proving is not restricted to the two-column proofs of Euclidean geometry, but is critical to creating and communicating ideas in all branches of mathematics (Knuth 2002). Harel and Sowder (1998: 241) define proving as the primary way "to remove or create doubts about the truth of an observation." They distinguish between two components: *Ascertaining* is an attempt to convince one's self; and *persuading* is an attempt to convince others. Ascertaining often has elements of exploration and insight, while persuading has influences of communication, argumentation, and perspective taking (Nathan 2013).

Investigators have looked at the role that directed actions and verbal hints play in fostering analytical proofs, specifically, transformational proof schemes, which "involve transformations of images--perhaps expressed in verbal or written statements--by means of deduction" (Harel & Sowder 1998: 258). Walkington and colleagues (2012; 2013; Williams et al. 2012) invited undergraduates to touch places on an interactive whiteboard in one of two conditions, either grounding actions, or non-grounding actions. Participants then assessed the truth of a conjecture and verbalized proofs and justifications for their reasoning. For the Triangle Task, the conjecture was

> *For any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side.*

Participants randomly assigned to the grounding actions condition touched symmetrically positioned spots on the board (scaled to each individual's arm spans) simultaneously with both hands, and as they did, new spots appeared that were further apart, eventually extending beyond their reach, and in doing so, bringing participants' chests in contact with the whiteboard (Figure 2). In doing these actions, participants essentially made a series of triangles with their arms and the board, which eventually degenerated into parallel lines when the spots exceeded their reach, and the distance from the board to their chest (the altitude of the triangle) was reduced to zero. Those in the non-grounding actions condition performed the same actions, touching the same places with equal frequency, but did not do so with both hands simultaneously, so they never experienced physically embodying the triangles. Though participants reported no awareness of the actions to the conjecture (following Thomas & Lleras 2007), those in the grounding actions condition were more likely to generate a key mathematical insight for the conjecture than participants in the non-grounding actions condition.

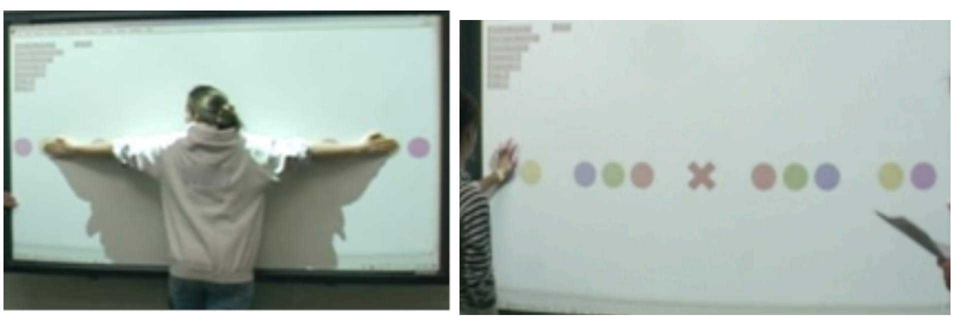

Figure 2. Participants perform relevant or irrelevant actions for the Triangle Task.

Although actions helped generate the insights related to ascertaining, the action condition by itself did not lead to differences in participants' abilities to *persuade*, or generate a mathematically correct proof. Participants in the grounding actions condition who received the hint were more likely to generate a mathematically correct proof than those with no hint, even though the hint merely mentioned in general terms that the prior actions could be relevant to the conjecture.

Promoting the insights involved in ascertaining truth for one's self appears to be influenced by one's actions, even when the relevance of the actions occur outside of one's awareness. Persuasion is more socially directed and must adhere to mathematical conventions.

Grounding actions are insufficient to promote mathematically correct persuasion without pedagogical directions to connect one's actions to one's reasoning. In this way, proof practices reveal some of ways that action and language together ground complex MR.

III. GROUNDED MATHEMATICAL COGNITION

This survey of findings from behavioral and biological studies of a broad set of mathematical phenomena observed along various developmental points, including members of post-industrial and indigenous cultural groups substantiates the view that MR is grounded through body states and actions, spatial systems, and language (Table 1). In support of the more constrained views of *embodied cognition*, body-based actions are indicated in tasks involving approximate and exact number sense, small number arithmetic, algebra equations, and the ascertaining phase of geometric proof. Evidence of the importance of spatial systems in these areas of MR are also well established, most notably the presence of the SNARC effect, which shows that the interpretation of numerical symbols are arranged along a number line of increasing values from left to right (for cultures where reading is left to right), and lateralized along the body's left-right axis. Language as a source of grounding is evident across the range of MR tasks as well, even in seemingly non-communicative activities, such as mental arithmetic. Language, of course, cannot be separated from its cultural home (Vygotsky 1978), and so culture, too, is indicated as a basis for grounded mathematical cognition.

With these findings in place, it is worthwhile to explore several broad themes. **First**, is to consider the MR literature in terms of Shapiro's (2010) distinction between the types of influence grounding has on cognition. *Conceptualization* addresses how mental events acquire meaning, and how those meanings come to be understood. Conceptualization has its closest parallels with linguistic determinism. Examples from cross cultural studies of the Pirahã (Gordon 2004) and the Munduruku´ (Pica et al. 2004) show support that the system of grounding influences the type and accuracy of MR. Cross-linguistic studies, such as that with the bilingual Russians (Dehaene et al. 1999), show that conceptualization is as viable in post-industrial cultures as it is in indigenous cultures. *Replacement* holds that the system of grounding can serve in place of some imputed mental representational system. Evidence from ethnographies of the Oksapmin (Saxe 1981; 1982) demonstrate that locations on the body can serve as the basis of an ordered system capable of supporting MR, which can be appropriated to new systems of quantification, such as learning a monetary system. *Constitution* posits that the processes for grounding cognition, the body-based actions and language-based ideas, may both cause and actually *be* the cognitive behaviors. The Constitution Hypothesis has its strongest supporters among adherents of situated and distributed cognition. Historically, cognitive science research has neglected those aspects of MR that are considered mathematical *practices*, emphasizing instead mental operations on mathematical content. As the view of MR expands to include the disciplinary behaviors of professionals as they actually do mathematics, support for the Constitution Hypothesis will inevitably grow, as these practices and the intellectual processes that enable them naturally go hand in hand.

The **second** issue is to explore how grounded MR can be realized. The Gesture as Simulated Action framework (GSA; Hostetter & Alibali 2008) hypothesizes that gestures arise during speaking when pre-motor activation, formed in response to motor or perceptual imagery, is activated beyond a speaker's current gesture inhibition threshold. GSA provides an account of how embodied actions arise as an outgrowth of cognitive activity, and suggests how recurring experiences may re-invoke those actions. The Gesture as Model Enactment (GAME; Nathan & Johnson 2011) builds on GSA, and proposes that actions coming into the system activate a feedforward network to generate multiple instances of likely cognitive states, where the most likely state is selected as further evidence is accumulated by tracking movements in relation to the environment and one's goals. GAME also draws on the HMOSAIC architecture of motor system control (Wolpert et al. 2003) and the theory proposed by Glenberg & Gallese (2012) that meaning in language derives from simulated actions drawn from the same systems that mediate hierarchical, goal-directed (i.e., planned) body movement.

One way to conceptualize these systems that couple motor control with cognitive processes is as *cognition-action transduction.* Transduction is a principle in the physical sciences that accounts for the transformation of one form of energy into another. Familiar transduction devices include electromagnetic motors (which convert electricity to mechanical motion), microphones (acoustic energy into electricity), and LEDs (electricity into light). Transduction devices exhibit an interesting property: They can run in either direction. Applying a force to a motor (reaching in and physically turning a rotor in a blender, for example), will output electrical energy, effectively becoming a generator. Likewise, light passing over an LED generates an electrical current, and effectively operates as a photodetector. The use of grounding action for promoting proof practices in geometry (Walkington et al. 2012, 2013; Williams et al. 2012) shows the potential of this framework for designing new types of interventions to promote conceptual learning in advanced MR through cognition-action transduction.

The **third** topic is to explore the implications grounded MR has for mathematics education. Identifying activities and curriculum materials that are commensurate with the kind of intellectual processing that mediates MR is a natural recommendation. Methods that support learners in performing the right kinds of actions are seen as positive steps for promoting the development of MR practices and concepts. Examples of effective interventions such as RightStart and Number Worlds (Griffin et al. 1994; 2004) are probably successful because they promote the formation of an integrated system of quantification that organizes one-to-one correspondence with numerals, numerical displays, and number words, all with a mental number line. PDL (Martin & Schwartz, 2005) supports MR with fractions by promoting the coevolution of appropriate actions and interpretations through object rearrangement. Proof practices improve when grounding actions foster students' insights about the conjecture, while hints enabled students to relate these body actions to language based argumentation, thereby formulating persuasive justifications (Walkington et al. 2012; 2013; Williams et al. 2012). Growth in support of theories that posit a reciprocal relation between action and cognition (e.g., Andres et al. 2007; Martin 2009; Nathan 2013) suggest pedagogical alternatives to transmitting propositional information through reading and direct instruction. Yet making overt connections and language-based redescriptions seems necessary to extend tacit insights into a form that will stand up to scrutiny. In this way, grounded MR can enable the individual to be an engaged participant within the community of mathematical thinkers and practitioners; and, like Galileo, foster one's appreciation of the language of the universe.

BIOGRAPHICAL NOTE

Mitchell J. Nathan, Professor of Educational Psychology, Curriculum & Instruction, and Psychology, University of Wisconsin-Madison, USA. Dr. Nathan's research is rooted in cognitive, embodied and socio-cultural perspectives on learning and classroom instruction in STEM fields (science, technology, engineering and mathematics).

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