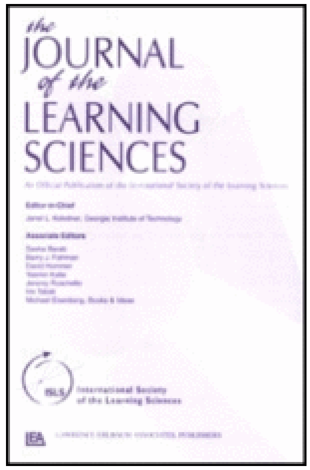

Threading Mathematics Through Symbols, Sketches, Software, Silicone and Wood: Tailoring High School STEM Instruction to Produce and Maintain Cohesion

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Threading Mathematics Through Symbols, Sketches, Software, Silicone and Wood: Tailoring High School STEM Instruction to Produce and Maintain Cohesion

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Abstract

A fundamental challenge in STEM education is that learners must recognize the inter-relatedness of mathematics ideas across a broad range of material and representational forms, settings and social structures, and realize how concepts (such as a quadratic relation) encountered in one form (e.g., an equation) relate to those same concepts encountered elsewhere (e.g., the behavior of a fully functioning device, like a catapult). In short, learners must come to recognize the *cohesion* of concepts and practices in their project based learning environments. Across 3 cases in technical education courses in mechanical and electrical engineering and college preparatory geometry, we show that cohesion of these central concepts in the learning environment cannot be assumed and must actively be produced and maintained. We identify three central processes to promote cohesion: guiding attention and behavior around *ecological shifts* in the learning environment, *coordination* of ideas across different spaces and representational forms, and *projection* of ideas forward and backward in time. We provide evidence that teachers' cohesion production actions are at times intentionally used to promote understanding, and show occasions where these cohesion production moves foster learning by enabling the construction of inter-relationships in the learning environment that were not evident to STEM students.

Threading Mathematics Through Symbols, Sketches, Software, Silicone and Wood: Tailoring High School STEM Instruction

Students in STEM (science, technology, engineering and mathematics) classrooms encounter a wide range of ideas and practices, as well as specialized vocabulary and representational systems for expressing those ideas and practices. For example, students' activities while working in project-based engineering, science or math classrooms are distributed over a variety of notational systems, including equations and diagrams; digital media, such as software simulations and electronic circuits; raw materials such as metal, plastic and wood; and designed objects, tools and measurement instruments. Furthermore, these varied encounters involve a range of participatory structures, such as those that commonly occur in classroom lectures, computer lab work, small group work, machine shops, and so forth. Within this framework, a fundamental challenge in STEM education is that learners must recognize the inter-relatedness of ideas across a broad range of material and representational forms, settings and social structures, and realize how concepts (such as a quadratic relation) encountered in one form (e.g., an equation) relate to those same concepts encountered elsewhere (e.g., the behavior of a fully functioning device). In short, learners must come to recognize the *cohesion* of concepts (Graesser et al., 2004) and practices in their project based learning environments, and cohesion needs to be fostered when it is lacking.

However, students do not readily make such deep connections across different representational, material and social forms (Ozogul & Reisslein, 2011). For example, in high school engineering classes, many students struggle to integrate previously encountered math concepts, such as those from geometry, with engineering activities such as computer-aided design (CAD) and measurement (Nathan, Oliver, Prevost, Tran & Phelps, 2009).

Johri and Olds (2011) note that many essential skills in engineering arise out of engagements not only with algorithms and inscriptions, but with tools, materials, and other people, as well. To encapsulate the broad range of activities that make up STEM practices of individuals and as observed in the activity systems in which they study, work, and live, and to emphasize their sensory and action-based qualities, Hall & Nemirovsky (in press) propose to focus on ***modal engagements*** within physical, cultural and social settings for mathematical activity. Modal engagements are defined as "a way of participating in activity, with others, tools, and symbols" (Hall & Nemirovsky, in press, p. 5). This term emphasizes the interactive and multi-modal nature of students' socially mediated, situated encounters with inscriptions, tools and materials. Hall and Nemirovsky's framework is an embodied one, which posits that all cognition is based on socially embedded sensorimotor activity; in their view, there are no "pure" (i.e., amodal) symbols or concepts, stripped of any trace of their bodily origin or implementation. Instead, Hall and Nemirovsky propose that mathematical activity can be analyzed along several dimensions that address how participants—using their bodies and senses—are positioned in space and time, as well as in relation to tools, materials, symbols, and social interactions. This falls in line with contemporary theorists such as Hutchins (1995), Hall (1996), Gallese & Lakoff (2005), Noble and colleagues (Noble, Nemirovsky, Wright & Tierney, 2001), and Lave (1988, p. 1), who posits, "'Cognition' observed in everyday practice is distributed—stretched over, not divided among—mind, body, activity and culturally organized settings (which include other actors)." In this work, we utilize Hall and Nemirovsky's concept of modal engagements in considering the nature of cohesion across contexts and representational forms in STEM instruction and learning.

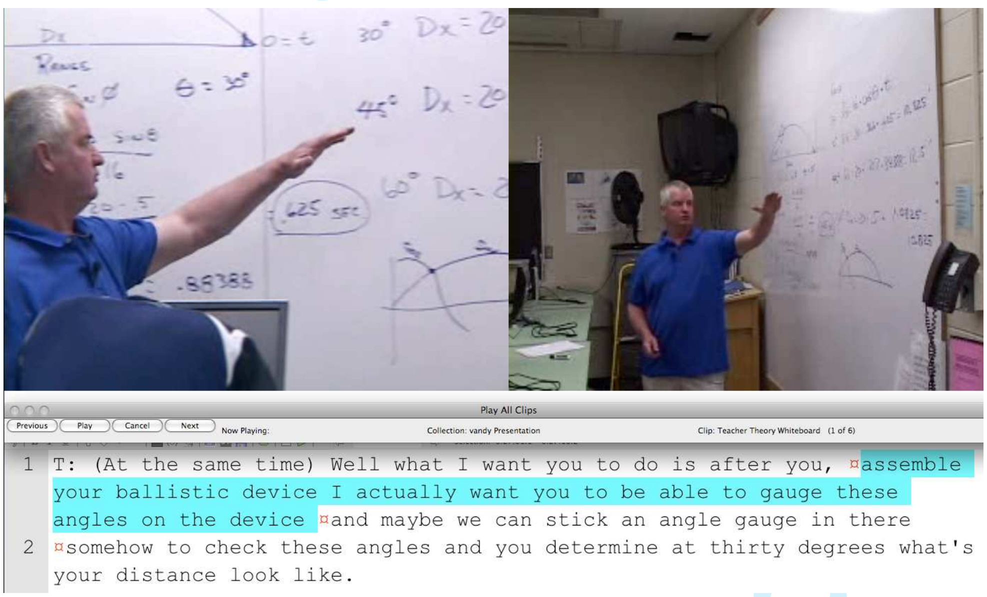
Though the need for integration across STEM disciplines is widely acknowledged (NRC, 2011), there is little systematic study of the challenges students face in finding cohesion across the many representational and material forms and participatory structures typically encountered during mathematics or science lessons. Classroom instruction and peer collaboration can at times foster the integration of mathematical ideas across STEM fields (Fairweather, 2008; Prevost, Nathan, Stein, Tran & Phelps, 2009), providing cohesiveness where the curricula do not.

Establishing Cohesion in the Classroom

Locating the Mathematics in Various People, Places and Things: The Where *of Mathematics*

In accounting for the continuity of mathematical ideas across such a broad range of STEM activities and education settings, we address two distinct but interrelated objectives. First, we identify efforts to establish cohesion in STEM learning environments that may aid students as they face discontinuities in their learning settings, such as shifts in activities, representation systems and contexts. Second, we investigate occasions of classroom learning when cohesion is produced and maintained. In this paper, we examine three cases drawn from high school mathematics and pre-college engineering courses to illustrate how STEM teachers and students manage cohesion in the classroom and shape the learning experience.

Consider, as one of our three examples, a typical engineering unit on building ballistic devices to hurl an object (a ping pong ball) at a target at an unspecified distance (Figure 1). During this multiday lesson students need to individually follow a lecture on the physical laws governing projectile motion expressed in algebra, geometry and trigonometry; collaboratively create, critique and revise 2D design sketches; use materials, measuring instruments and tools (both handheld and power tools) to construct the device within teams; compile measurements during field testing and analyze the data back in the lab; and so on. Each phase calls for one or more material forms, representations or tools to take center stage; each may be scheduled in a different space (wood shop, classroom) and social configuration (lecture, small groups); and each has a historical relationship with the events it follows and precedes (Latour, 1996). Through it all, cohesion of common math concepts that describe the intended trajectory of the projectile must be obtained. By identifying *where the mathematics is located,* we establish its existence and describe the actions teachers and students use to maintain that cohesion across its many manifestations.



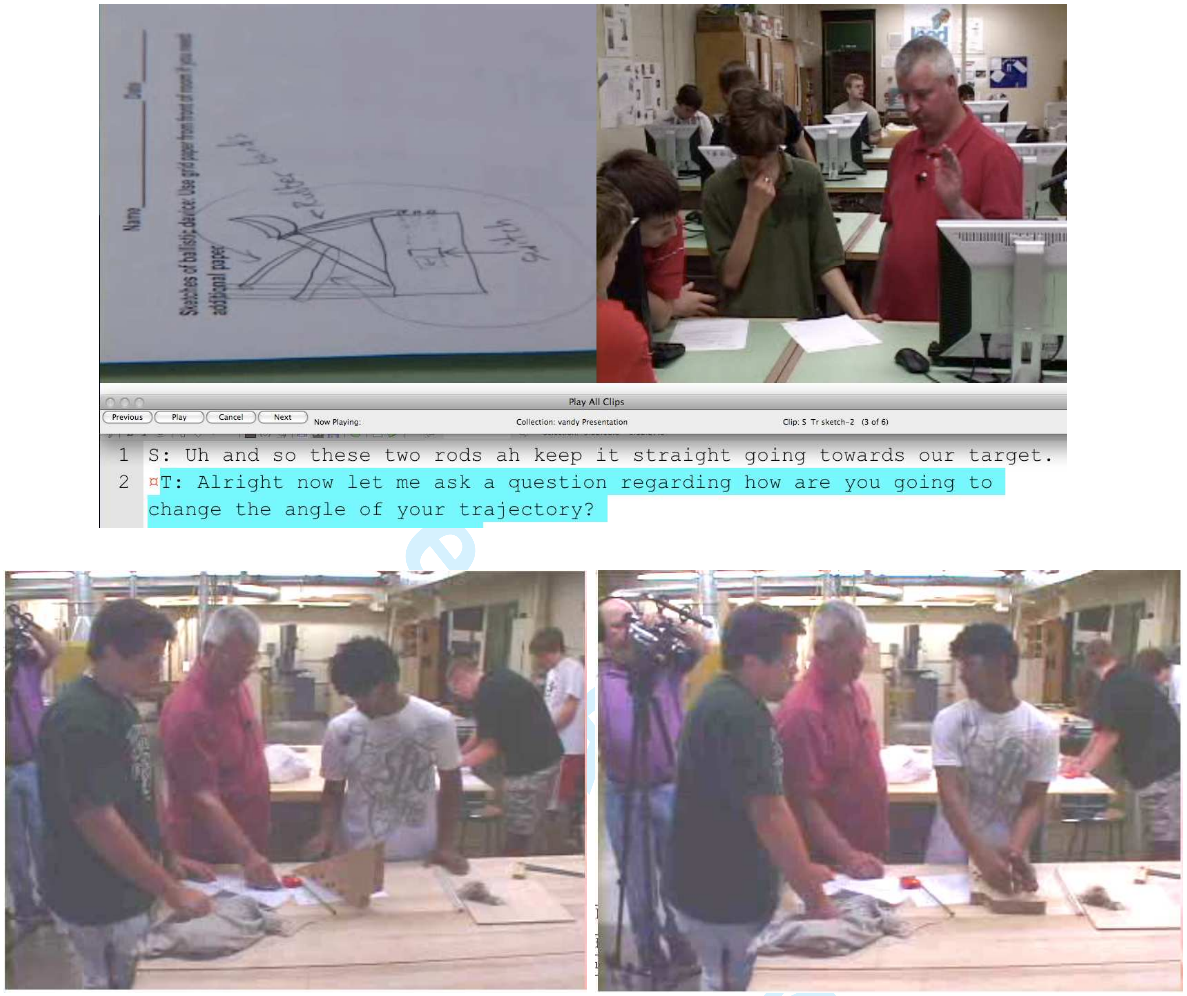


Figure 1. (a) (Top 2 panels) The mathematics and physics of kinematics that model ballistic motion must also be connected to (b) the 2D design sketch (middle 2 panels), and (c) the construction, testing, and redesign of the ballistic device (bottom 2 panels). Note that the teacher attempts to connect the design sketch to the wood in the construction phase (left panel), but the student focuses on the wood, to the exclusion of any cross-modal connections (right panel).

Transitions Across Activities, Representations and Settings

The processes by which teachers and students manage the transitions across curricular activities and settings while maintaining cohesion of central concepts are both complex and precarious. We have identified three ways that transitions across activities, representations and settings are managed by teachers and students. One is that the participants make an ***ecological shift,*** a reorientation of the activity context that can include different spaces, tools, instructional media, and participant structures. Ecological shifts often introduce changes of the spaces and social structures in which participants operate. At its surface, a shift can simply appear as a change in the activity, as, for example, when a geometry teacher called the class to stop their computer lab work and focus attention on her. Alternatively, the shift may be more momentous, as when the engineering teacher took his students from the classroom, down the hall, to the wood shop, which altered the norms of proper (i.e., safe) conduct, the resources at their disposal, ambient sounds, and the participant structures, while also placing what had previously been planned for the future into the present task of implementing the proposed designs. Ecological shifts can profoundly alter the social and physical environment and the available resources.

A second transition process, ***projection***, involves the use of multimodal language to connect events of the present to past or future modal engagements. Past projections can link across an ecological shift that has already occurred, while future projections can anticipate a coming shift in the classroom ecology (one that may even be part of the curricular design). Projections can take many forms (cf. Engle, 2006). Some are brief utterances, as when a teacher references the activities of the previous day's lab, or simple pointing gestures, as when a teacher points to an empty white board to re-invoke the mathematical derivations from a prior lesson. Others are much more protracted, as when a teacher spends an entire introductory lesson planning the lab work for the rest of the week. Teachers and students use the verbalizations and gestures of projection, along with representations, objects, and the environment itself, both to reflect upon the history of a concept as it unfolds in their classroom, and to plan for future manifestations of the concept in different modal engagements.

A third transition process is ***coordination***, which involves the juxtaposition and linking of different material and representational forms, following the related use of the term by diSessa (1991) and others (e.g., Hutchins, 1995; Stevens & Hall, 1998) who describe coordination across agents, physical objects, and representations. For example, students may coordinate a design created in a software environment with an actual device they are building, or they may enact coordination between symbolic and tabular representations of the logic of a digital circuit. When speakers integrate across time and material or representational forms simultaneously -- as when they make a connection from a device in their immediate context to a previously encountered but now absent equation -- we consider this both coordination and projection.

Ecological shifts – common as they appear to be – can make it challenging for participants to preserve a sense of the cohesion of the learning environment. Projection and coordination serve to establish cohesion over time, across interactions, and throughout the current environment.

Identifying Locally Invariant Relations: The* What *of Mathematics

In addition to identifying *where* the math is located and describing transitions across representations, settings and social configurations, it is important to be able to say ***what** is the mathematics* across shifting social configurations, physical settings and material and symbolic forms. We have found that the stability of the mathematical content across contexts and forms is something that has to be produced and maintained "locally" by the agents.

A complete answer to the question of "What is mathematics, really?" is elusive (cf. Hersh, 1997; Lakoff & Nuñez, 2000; Newman, 1956/2000). To Aristotle, mathematical objects are physical objects considered in a particular, abstract way, and they transcend any particular manifestation (e.g., Aristotle, 1984; Lear, 1982; Mignucci, 1987). From this perspective, mathematics concepts are invariant across contexts, agents and forms, even when some or all of the outward qualities have changed. Under the "romantic view" (summarized by Lakoff and Nunez, 2000), mathematics is captured in the formalisms that represent it, and its meaning derives solely from its representational structure (cf. The Bourbaki Group, 1950). These positions marginalize social and cultural influences and the manner in which mathematical ideas are archived, interpreted, taught and applied (Hersh, 1997).

An alternative position (Noble, Nemirovsky, Wright & Tierney, 2001) characterizes mathematical concepts in terms of what people *do* rather than in terms of their *form*-al qualities (i.e., in terms of their shape and syntactic form, as manifested in symbolic or material entities). From this perspective, cohesion of meaning across modal engagements comes from the many inter-relations between forms and experiences that share (sometimes implicit) characteristics and differ in others, just as "the strength of the thread does not reside in the fact that some one fibre runs through its whole length, but in the overlapping of many fibres" (Wittgenstein, 1958, p. 32).

For our purposes, these views of mathematical concepts all appear to be insufficient for framing the study of Western science classrooms. One reason is the normative nature of mathematics education. In classrooms the authoritative voices from the course curriculum and the instructor assume the existence of invariant properties tied to specific concepts, and these concepts must be learned and applied during high-stakes assessments to satisfy state and national objectives. There may be powerful philosophical arguments regarding why these relations do not really exist, are not universal, or do not hold as a matter of necessity. But there seems to also be a need to acknowledge that in highly constrained circumstances -- specific relations in specific contexts brought forth to achieve specific curricular goals -- trained practitioners in STEM fields can reliably identify ***locally invariant*** qualities of ideas even as they transition across representations and contexts (Appendix A). Like the temptation to declare a “flat Earth” when constructing a house, activities sufficiently localized in space and time can provide mathematical experiences that sufficiently approximate the invariant structures posited by the essentialists, but with an understanding that the dynamics of these modal engagements are central to the mathematical experiences of students and ultimately influence how students represent and enact their experience-based knowledge.

An example of the social production of a locally invariant relation comes from the projectile motion unit for the high school engineering class we introduced earlier, where there is a need to characterize *theta*, the angle of ascent, across a range of modal engagements (Figure 1). In the classroom, the teacher and the students work to represent this mathematical relation in several ways; a raised arm to the base of a triangle, a Greek symbol, a numeric measure, a tangent line meeting a plane, and the relation between the trajectory of an object and the ground, as *theta* is realized, respectively, by the flight of a ball, a lecture, an equation, a sextant, or an idealized diagram in analytic geometry. By focusing on *relations* as the *what* of mathematics we direct our efforts at understanding how mathematical values are communicated and realized in situated representations, in the functioning of constructed devices, and in students’ multimodal interactions with their social groups, objects and inscriptions.

Thus, we assume that what a community regards as locally invariant in each manifestation of the mathematics is some kind of central *relationship*. Following Hall and Nemirovsky (in press), this does not mean that the mathematical concepts of concern are *amodal*; as we will illustrate, the mathematical concepts as they are experienced and practiced by teachers and students are highly subject to the modal engagements in which they arise. Yet, there are common relations that can be analogically or metaphorically mapped from one modal engagement to another (Gentner & Markman, 1997; Lakoff & Johnson, 1980). People performing a mathematical activity can perceive, maintain, and even construct locally invariant relations by using relation- and inference-preserving cognitive mechanisms such as analogical mapping and conceptual metaphor. In this way, participants actively build and maintain cohesion of locally invariant relations across representations and shifting contexts.

The Nature of Learning within a Cohesion-Production Perspective

A commitment to cohesion suggests some specific ways that we can expect to observe learning in the classroom. One form of learning is when students develop a new awareness of the existence of an invariant relation. Development of one's "disciplined perception" is evident when people "learn to see with and through their inscriptions" (Stevens & Hall, 1998, p. 108). For example, in their analysis of a one-on-one tutoring session on linear functions, Stevens and Hall (1998) showed how an eighth grade student's pre-existing ways of "seeing" linear relations in terms of a grid view of the Cartesian coordinate system changed to include those relations in equation form, thereby permitting the student to engage in a vast new range of mathematical activities. Change, they argued, was brought about through acts of "disciplining perception," in which interactions with the tutor ("look at it this way") over increasingly less hospitable tasks reorganized the student's visual orientation to the mathematical inscriptions.

Learning is also exhibited when students demonstrate new connections that cross distinct modal engagements. These connections are especially notable when they alter the affordances (Gibson, 1979) of modal engagements, so that they are seen and managed differently by virtue of their new associations. Nathan and colleagues (2011), for example, showed that engineering students initially had difficulty mapping Boolean logic onto actual digital “chips” needed to wire a circuit and implement a logic model. They found that students struggled with the high degree of visual symmetry of the chips, which belied the true asymmetries of the chip’s arrangement of power, ground and the inputs and outputs of the logic gates. One team of students reified the coordination of the chip locations with the schematic layout using color-coded wires. This allowed students to perceive the previously hidden arrangement of the logic circuit, which led to changes in how they used the chips and in the operations afforded by them.

Focus of Research

From our perspective, the cohesion of locally invariant relations across diverse modal engagements cannot be assumed in STEM curricula, and the means for creating such cohesion are neither obvious nor universal. Novices operating in STEM classrooms and workplace environments have seldom been challenged to construct for themselves the deep mathematical understandings and broad connections that would allow them to notice invariant relations across the various modal forms. Consequently, novices need to be socialized into perceiving the same invariants that are salient to experts (Stevens & Hall, 1998). Thus, we set out to show that the cohesion of mathematical knowledge across contexts is something that has to be produced and enforced locally by the participants. We provide evidence that teachers’ actions are at times intentionally directed at cohesion production. Furthermore, we demonstrate that cohesion production can foster learning by highlighting the existence of locally invariant relations that were previously overlooked by students and by making new connections that alter the affordances of modal engagements. We posit that many features of curriculum and instruction in STEM education exist in order to highlight relations, with the goal of advancing students’ perceptions of locally invariant properties so that they serve as a cohesive thread throughout the STEM activities.

The analysis that follows focuses on how teachers and students establish cohesion of locally invariant mathematical relations (the *what* of mathematics) as the relations are projected and coordinated across various modal engagements (the *where*). As our first research question, we ask: **How is cohesion of core mathematical ideas (relations) produced and maintained in STEM classrooms?** To address this we investigate specific means by which continuity of locally invariant relations is signaled as participants operate across various activities, social configurations and notational systems. We expect that greater cohesion in the learning environment makes possible new sensitivities that may lead to transformations of students' perceptions and actions. Therefore, as our second research question, we ask: **How does cohesion production support learning?** Lastly, we explore whether teachers are aware of the need to help students to perceive continuity across the range of modal engagements, and whether teachers intentionally design or regulate their instruction to foster greater cohesion. We ask: **How are teachers' instructional moves shaped by the need to establish and maintain a cohesive learning environment?** We explore these questions in three classroom cases. In the final section we consider the implications of this work, and in particular, how it informs us about three core issues: the structure of normative mathematics, the nature of learning and instruction in STEM classrooms, and the challenges facing teachers and curriculum developers as they seek to design and implement learning environments that foster STEM integration.

Theoretical Framework: Cohesion Production and Maintenance

Our first research question focuses on how mathematical ideas (relations) are realized within and across modal engagements as they occur in STEM classrooms. To address this question, we focus on both the *where* and the *what* of mathematics. In addressing the *where* of mathematics, we consider how invariant mathematical relations are realized in a given modal engagement across forms, time and space, and we consider the affordances and constraints that each modal engagement exhibits for reasoning mathematically. We examine a broad range of behaviors and contexts, including social interactions in instruction (lectures, coaching) and student discussions (investigations, explanations and elaborations, questions, design decisions); artifacts (designs, tools, devices) and symbol systems (language, symbolic, algorithmic and visual representations). We also consider changes in the learning environment, and the development of ideas and practices over time, including the history, present focus and future planning across various phases of a classroom unit.

We also address the *what* of mathematics, by exploring how mathematical relations can be preserved when they are manifest in markedly different ways. Our analysis foregrounds the mathematical relations that are deemed by STEM experts to be locally invariant across modal engagements regardless of their outward form. This investigation raises important questions and insights about how mathematics both facilitates and obfuscates the integration of concepts for learners across scientific fields and phases of project based learning activities.

To foreshadow, we find that these connections are not readily apparent to students, so the teacher and the students must continually manage and negotiate the establishment of cohesion, and to do so they rely heavily on language, gesture, and other forms of visual scaffolding. Speech provides cohesion by using resources such as labels and explanations. As will be made clear from the cases below, however, simply referring to mathematical ideas using consistent labels across different contexts is not sufficient for most students to establish the cohesion necessary to complete their projects and to develop a clear understanding of how the mathematics permeates the various activities and representations. Along with speech, teachers also use gestures to establish and maintain cohesion. Gestures provide cohesion by connecting related ideas and/or visual representations (Alibali & Nathan, in press; Alibali, Nathan, & Fujimori, 2011; McNeill & Duncan, 2000; Nathan, 2008; Williams, in press). Teachers also provide other forms of visual signaling (Ozogul & Reisslein, 2011), including written inscriptions such as equations, diagrams and words that reify concepts, relationships and plans in a manner that is (relatively) enduring, and that highlight relationships and connections.

Our second research question concerns how cohesion production supports learning. In addressing this question, we show how the perception of invariant mathematical relations is socialized in STEM classrooms, and we look for evidence in students' behaviors that they have come to perceive and understand these invariant relations. Initially, for students, the locally invariant properties of a particular mathematical concept may not rise above the din of variation across material and representational forms. If, as we hypothesize, cohesion production supports learning, then we expect to see changes in student actions and explanations. In particular, we may see students acting in ways that are attuned to their developing understandings.

Our third research question focuses on how teachers' instructional moves are shaped by their perceived need to establish and maintain cohesion in the learning environment. To address this question, we explore teachers' intentional uses of speech, gesture, and body-based and environmental resources, both for establishing cohesion and for identifying breaks in cohesion that can be disruptive for students. We draw on pre- and post-lesson interviews to understand how teachers strive to restructure the environment to focus attention on invariant relations and help to provide cohesion that that may enable students to thread together their classroom experiences and conceptual understandings.

Method

We conducted multiday observations and collected dual-camera videos of teaching and learning in each of three different high school courses: Principles of Engineering (mechanics, 4 days), Digital Electronics (logic circuit design, 4 days), and Euclidean Geometry (proof, 3 days). On their own, each case yields valuable insights about how mathematical ideas are maintained and connected in different modal engagements during instruction, activity and communicating. The cases also share some common elements. Comparisons between geometry and digital electronics highlight approaches to proof; mechanical engineering and geometry share spatial reasoning and geometric construction; and digital electronics and engineering share technical education and engineering design. In combination, these cases illustrate the ever-present need to maintain cohesion during transitions across a broad range of modal engagements, and they illustrate some of the ways cohesion can be supported by teachers, peers and curricula.

Videos of these lessons were transcribed in Transana, a platform that allows for the integrated viewing of multiple audio and video feeds (i.e., multiple camera angles) and transcripts. A research team comprised of mathematics and STEM education researchers, math teachers, a linguistic anthropologist and a cognitive developmental psychologist, including all the team members who observed the original classroom events, met regularly to collaboratively review the integrated video-transcripts. Over a series of several months, the review team identified invariant mathematical ideas and ecological shifts documented in the episodes. Patterns of interactions among class participants and modal engagements within the videotaped ecological contexts were proposed, and these patterns were investigated in during more focused coding sessions using the interactive functionality of Transana. From these data viewing and analysis sessions, the members of the research team identified segments of discourse within the lessons that manifested coordination, and projection, as described above and in Table 1.

Table 1. Coding criteria for the production and maintenance of cohesion

Transition	Coding Criteria
Ecological Shift	Evidence of a major reorientation of classroom activity to involve different settings, participation structures, representational and material forms, tools, or actions.
Projection	Evidence that participants refer to an absent (past, planned or imagined) modal engagement.
Coordination	Evidence that participants link two or more co-present material or representational forms.
Projection + Coordination	Evidence that participants make a projection to an absent engagement while also linking this to a currently present material or representation.

Findings from Three Case Studies

The three classrooms that we describe -- in which students engineer ballistics devices, build digital circuits for a secure voting booth, and explore properties of angles inscribed in circles -- illustrate how teachers and students manage the process of threading concepts through rich ecological contexts. We present descriptions of each of the three classrooms (with transcript excerpts) and discuss their connections to our emerging theory.

Theta in Symbols, Paper and Wood: A Ballistic Device Design in a Precollege Engineering Classroom on Mechanical Engineering

The ballistics project challenge

"I'm actually gonna give you a distance and I'm gonna say 'okay we're gonna send, we're gonna set the basket fifteen feet away,' but whatever distance that is I'm gonna decide that at the time, we're gonna set the basket so many feet away and you have to try to hit it. So by doing some calculations on, what youre, um, ballistic device fires you can kinda set your angle hopefully to get, to get that distance." (Principles of Engineering teacher, Day One).

On Day One of this lesson, the students in a second year pre-college engineering course learned the mathematics and physics of calculating projectile motion. The teacher highlighted for them the angle of ascent of the projectile—labeled *theta*—as the key variable that they needed to parameterize and represent in their sketches (one group's sketch is shown in Figure 2) and ultimately in the wood, metal, plastic and other materials that they fashioned and assembled into a catapult, trebuchet, gun or other ballistic device of their own choosing and design. If these devices properly instantiate *theta*—that is, permit the adjustment of the angle of release while holding constant the other influential variables (e.g., initial velocity)—students will be able to predict the distance that the projectile will travel. Throughout the sequence of the lesson, knowledge of *theta* is inscribed or represented in different modal forms: symbols and diagrams on the white board during an initial lecture, paper and pencil during small group design meetings, and collections of materials formed into projectile devices, which are ultimately manipulated and evaluated.

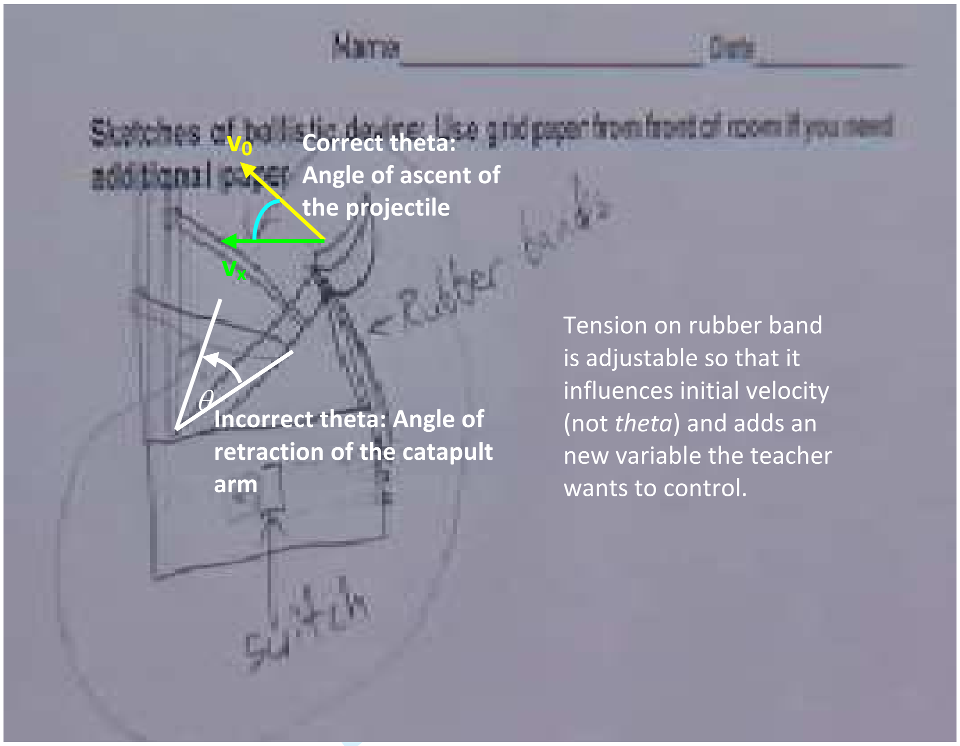


Figure 2. One group's design sketch (with verbal and mathematical elaborations added).

The events depicted in the transcript below took place after a group of students presented their sketch of a catapult to the teacher on Day 2 of the four-day unit. The discussion of the students' design is sandwiched between the more formal lecture on kinematics (including algebra, trigonometry and the idealized behavior of an object in free fall with a constant horizontal velocity) and the material construction activity. Many aspects of this discussion project toward the future context where the students will use their sketch to guide construction, and many aspects project back to a past lecture that presented kinematics laws and mathematical relations.

This case illustrates how easily students can lose sight of a central mathematical concept, and how this results in a breakdown that leads to poor engineering design. Over the course of the small group discussion, a breakdown of cohesion is apparent: These students have confused the angle of ascent of the projectile with another angle in the system -- the angle of retraction of the arm of the catapult (Figure 2). Despite the foregrounding from previous activities and despite the juxtaposition in time of the kinematics lecture and the design activity, it becomes evident that nowhere in the curriculum has the continuity of the core mathematical idea of *theta* been supported in this design project. Thus, the students' design sketch not only misidentifies the relevant angle (a failure to coordinate *theta* as described in the lecture with the design sketch) but it also introduces another variable -- the tension on the rubber band, which influences the initial velocity. The teacher attempts to point out this confusion in Lines 1 and 3 (See Transcript #1). The students in Lines 4-8 try to salvage their sketch. Based on the constrained use of their gestures and eye gaze, and restricted references used in their speech, the students are focused almost exclusively on the properties of the sketch itself (with some connections made between the drawing and the intended function and the dynamic nature of the lever arm; see Figure 1b), to the exclusion of the mathematical relations that model the object's ballistic behavior.

On Line 9 the teacher explains that the angle that they need to control is the angle of the ascent of the projectile with respect to the ground and, continuing on Line 11, that they need to design something that does not simultaneously affect the initial velocity. The tacit implication is that varying initial velocity introduces new complications that were not addressed in the mathematical models presented during the previous class. In Lines 12 through 15 a student defends their choice and in so doing further confirms that they are not considering the parameterized relation of distance traveled as a function of angle of ascent. In Lines 16 through 23 the teacher coordinates and projects the students' design sketch backward to the math relations presented the day before on the whiteboard (which are still present in the front of the room) and forward to the future behavior of the yet-to-be-realized device. This coordination is accomplished through speech and gesture. The first gesture on Line 18 is a flat-palmed hand lined up horizontally with the diagram, iconically representing *theta* as an angle relating the initial trajectory of the projectile with the ground, though translated into the plane of the paper sitting on the desk. This is an important reference because the hand shape and motion re-invoke a similar gesture—a gestural *catchment* (McNeill & Duncan, 2000)—that the teacher used during the lecture on the mathematics (algebra and trigonometry) of projectile motion. The parallel between the gesture used earlier at the white board and the one enacted here highlights the recurrent hand shape as an emerging "sign" for *theta*. The second gesture on Line 19 is a point indicating the calculations from a previous class that are still on the whiteboard. The teacher uses speech and gesture to coordinate the calculations on the board with the students' diagram in an effort to locate *theta* in the design sketch and reinstate its original meaning. The point appears to be taking hold as, for the first time during the discussion, one of the students (Line 22) acknowledges the relevance of the mathematical relations for their design. Yet as we also see from the still images (Lines 18 and 19), the students are fixated on their own work and give little attention to either the iconic angle gesture or the overt point to the whiteboard. The result is that little was taken up by these students, and their design remained largely unchanged.

Insert Case #1 Transcript 1 about here

By way of summary, we reflect on this case in the language of our emerging theoretical framework. The **locally invariant relation (the *what* of mathematics)** is the angle of ascent of a projectile with respect to the ground, as represented initially by the symbol *theta*, and the role it plays in predicting the distance traveled. The ***where* of mathematics** is described in terms of ecological shifts and transitions between modal engagements. To foster **cohesion**, we see the teacher **threading the mathematics through** the various modal engagements. The teacher uses speech and a gestural catchment to coordinate the angle *theta* and its meaning for projectile motion with the elements of the design sketch. He also identifies an important misconception, in which students improperly identify an element of the catapult design as an instantiation of *theta*, leading them to control the initial velocity rather than the angle of ascent with respect to the ground. Projection is used to signal for the student the historical role of the design sketch. Past projections are made to the mathematical formalisms that model projectile motion that were presented in the previous class. A future projection is intended to position the sketch as a guide for the construction activity awaiting the students. Here the teacher specifically refers to the impending testing of the device to clarify that they will not be changing the initial velocity but trying to keep that constant while varying the angle of release (*theta*) as a way to hit targets at varying distances. In these ways, *theta* serves as a central, invariant mathematical relation threaded through a range of settings, social exchanges, and material and symbolic forms.

The analysis of this case illustrates how challenging it can be to thread mathematics concepts through project activities. Figure 3 shows how the entire sequence of the ballistics project was coded for modal engagements and transitions. The analysis shows the hierarchical structure of the lessons (cf. Baker et al., 2008), in that modal engagements are nested within ecological contexts. Columns show the various ecological contexts in which the activities were embedded throughout the project. Arrows in the figure illustrate the roles of projection (italicized text) and coordination (underlined text) in the management of ecological shifts. In particular, the figure shows how the teacher often used backward (left arrow) and forward (right arrow) projections together to bridge present modal engagements (bulleted entries within each context), such as working with design sketches, to those in the past (e.g., the physics and mathematics of projectile motion) and those that will be used in the future. The figure also indicates more frequent use of forward projection as the teacher prepared students to build and test the ballistic device. The enactment of coordination and project are insufficient, however, if these are not attended to or taken up by learners. The students' fixations with their own work during these crucial moments thwart the teacher's cohesion-producing efforts. This first case, then, illustrates the need to foster cohesion in the modally rich learning environment and the challenges of doing so when students do not direct their full attention to the teacher's instructional moves.

As evidence that this teacher was sensitive to the challenges students face in grasping the cohesion of ideas across modal engagements, during the post-lesson interview from Day 2, the teacher discussed the backwards projections he made to the kinematics lecture while assisting groups with their sketches (e.g., lines 16-23).

> [I]n some cases [the designs] were too simple and and not very complete in that sense that they didn't really indicate to me what the, how they would do that. How are they gonna change their angle. How are they gonna sh- show what the angles are. So I was looking for something and I related it to the fact that, you know, we talked about it yesterday that we were going to have to propel this ball to a certain distance... [I]n reference to the work we did yesterday, they could-- we could see that the angle of the trajectory is going to affect the distance that they are able to shoot it.

The teacher was quite explicit about the connections students needed to make between the math and physics presented the previous day and the nature of the design sketch (Lines 16-19). In critiquing students' designs, the teacher wanted them to recognize a once-marginal aspect of the sketch (the angle of ascent as one of many angles in the device) as a key parameter of the system. In this way, the sketches did not merely resemble devices to be made, but they properly modeled central kinematic relations of a working ballistics device.

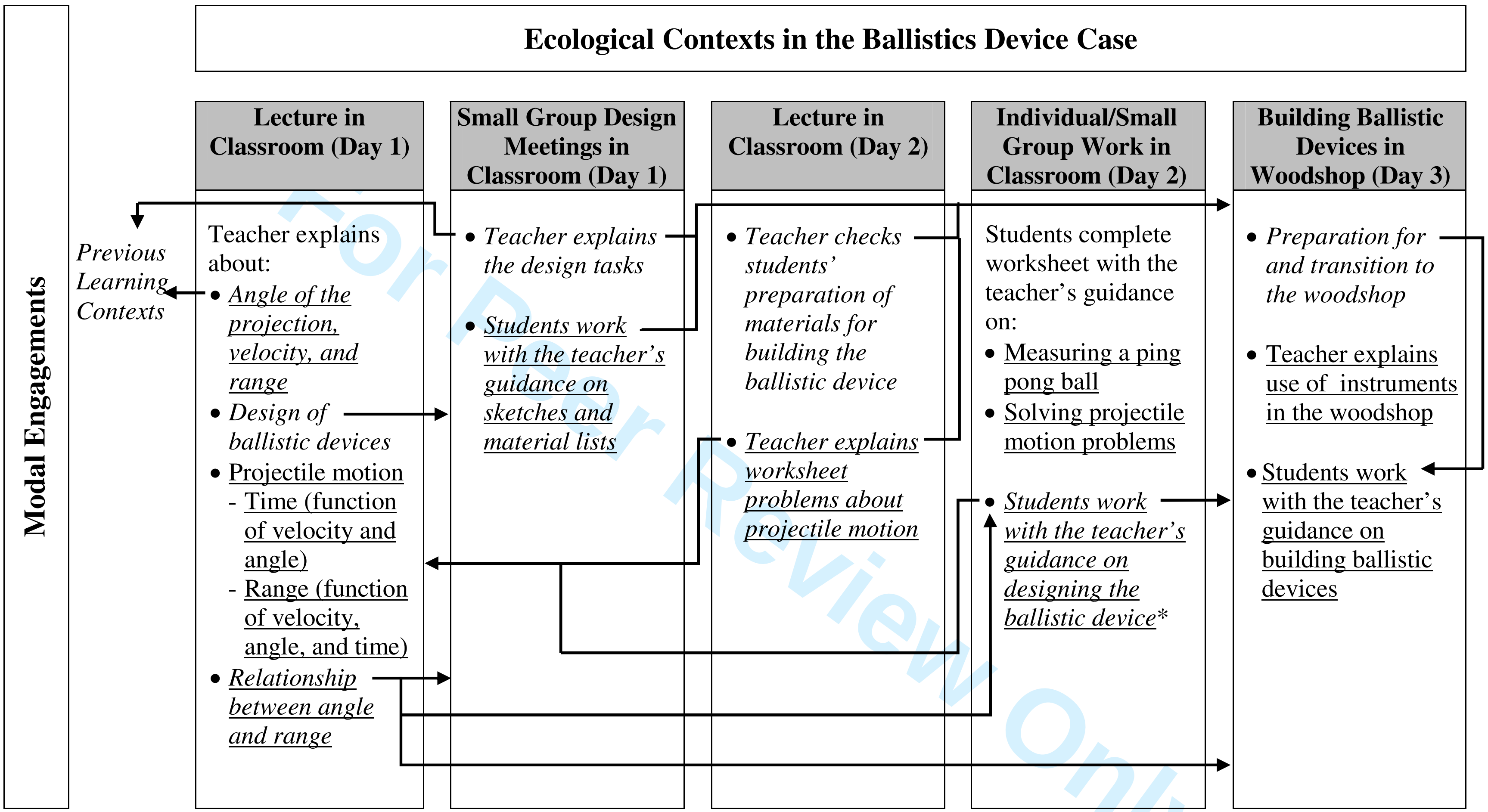


Figure 3. Boxes show the major ecological contexts that activity was embedded in throughout the ballistic device case. Bullets show the main modal engagements occurring sequentially in the case. *Italics* = *Projection*, Underline = Coordination, *Italics and Underline* = *Projection + Coordination*. Arrows show the main backward/forward projections with wedge-shaped ends pointing to projected past/future modal engagement(s). *indicates the modal engagement discussed in the transcript of the ballistic device case.

Logic Enacted Through Boolean Algebra, Simulation and Silicone: Designing a Secure Digital Voting Booth

The main activity of this digital electronics lesson was to design a voting booth privacy monitoring system. An effective monitoring circuit is indicated by two outputs: a green light-emitting diode (LED) that is activated whenever a particular voting booth is available for use, and a red LED that lights up whenever privacy is at risk and entry is denied.

The circuit design involved implementing the basic set of logical constraints and conditions into a working electronic circuit that outputs a green light when all of the conditions are met, or a red light (alarm) when any condition is violated. The process unfolds across the following activities: verbally introducing the problem ("For privacy reasons, a voting booth can only be used if the booth on either side is unoccupied."), along with a "block diagram" representing the monitoring system, and an equipment list; discussing a completed truth table with entries composed of 1's and 0's accounting for all of the possible states of the circuit (voting booth occupancy and LED output) and a related, spatial Karnaugh map (K-map); generating and manipulating a set of Boolean algebraic expressions consistent with the K-map; drawing an Automated Optical Inspection (AOI) circuit; modeling the circuit in the MultiSims software to create computer generated SIM diagrams; and building and debugging a working electronic circuit made of a "bread board," integrated circuits, resistors and capacitors, wires, a power source and LEDs. Similar to the projectile motion lesson, the mathematics in this lesson (here Boolean logic rather than the algebra and trigonometry of kinematics) is manifest through a sequence of modal engagements with instructional contexts, representations and a range of material forms traversed by ecological shifts. The teacher often sought to establish cohesion between the relations modeled by the Boolean algebra and different forms of materials and representations by using coordination and by projecting toward past or future modal engagements throughout the four-day lesson.

As a practical matter, to use the integrated circuits as a source for specific logic operations (e.g., AND, NOT, OR), the teacher and students needed to consult something referred to as the "data sheet," which was a set of documents affixed on a poster board. This printed material illustrated the formal specifications of different logic gates and their layouts in each integrated circuit, which varied by manufacturer. The data sheets established the connections between idealized logic symbols for Boolean operations such as AND, NOT and OR, and the actual locations of the inputs and outputs of specific integrated circuit components. Furthermore the computer based MultiSim diagrams are tied to and constrained by the layout of the specific chips as determined by the manufacturer. Unlike the symbolic inscriptions (i.e., the algebra and truth tables), the MultiSim diagram (see Figure 4) spatializes the logical relations by imposing spatial locations of each of the logical operations and determining the particular paths of information flow, with input-to-output relations generally moving from left to right.

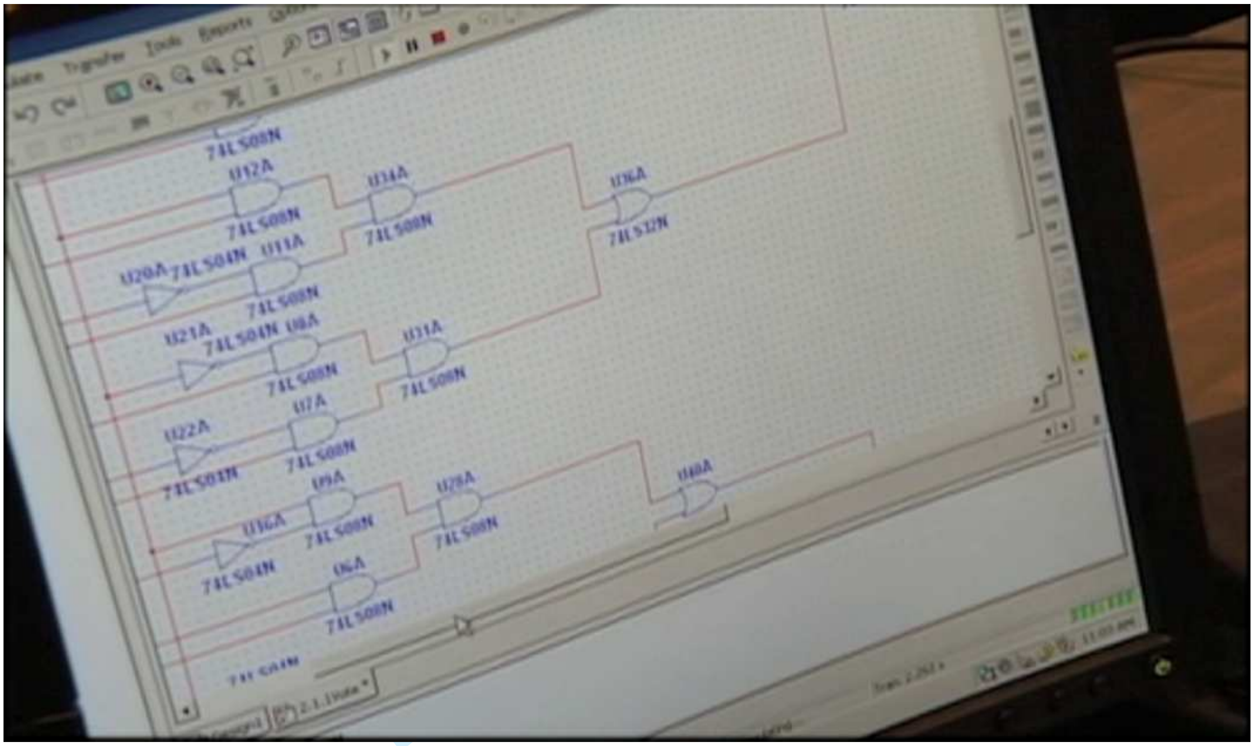


Figure 4. A sample MultiSim diagram. Chip model numbers are used to label the various logic gates in the digital schematic.

Although there is an identifiable local invariant relation for the logic of the voting booth monitoring system, here, as with the trigonometric invariant in the ballistics case, students' understanding the cohesion of the multiple phases of this project has not previously been developed. Our analysis of the digital electronics case study illustrates several transition processes and the coordination between various modal engagements.

The following transcript is an excerpt from the last observed day of the lesson in which the teacher initiates an ecological shift (Line 1) by calling all students to gather at the lab station of the student group that had made the most progress on their voting booth monitor. The class witnessed the conversation between the teacher and a member of the group about checking the circuit for accuracy and discussing how the circuit may be improved. The teacher starts out at the end of Line 1 forecasting comments he wants to make about the organization of the wires in the board. Anticipating this, (Line 2), the student acknowledges, “it’s messy. I get it.” Regardless, in Lines 3 through 11 the teacher addresses this practical matter that is not evident in the symbolic or simulation-based representations—the need for an orderly and “clean” wiring job (“it’s just the spaghetti mess” in Line 15).

The teacher in Line 16 then starts to model how to establish coordination of the physical arrangement of wires, integrated circuits (“chips”), and electronic components using speech and gesture with the simulated circuit shown in the SIM diagram. In Line 17, the student takes up the teacher’s troubleshooting practice. However, the student chooses to work from the truth table rather than the SIM diagram. Practically speaking, this allows the student to turn each input switch to the circuit on and off to model the occupancy state of the voting booth (ON = Occupied, OFF = Vacant). By mapping the entries in the truth table directly to the circuit, the student bypasses the conceptual connection of the circuit to the Boolean expressions that are central to the SIM representation and to the original problem context. The more narrow set of associations selected by the student exemplifies the challenges that students and teachers regularly face in constructing and maintaining cohesion of the concepts across the many forms in which they are manifest in complex, multimodal projects.

The dialogue from Lines 18-24 and the corresponding gestures show the teacher’s troubleshooting method using the systematic coordination of the entries in the truth table with the state of the digital circuit on the breadboard. The teacher models how each entry in the truth table maps directly to a physical state of the circuit, running his finger from one row of the table to the next. Coordination between the table and the circuit provides situationally relevant feedback (the state of the green and red LEDs), and at the same time establishes the meaning of the symbolic table entries (cf. Nathan, Kintsch & Young, 1992).

Initially (Line 22), the teacher calls out the circuit inputs, while the student sets the switches appropriately ("zero, zero, zero" means all of the switches are in the OFF position since none of the booths are occupied). The student echoes the teacher in Line 23, reporting on the state of the input switches, and then describes the output (e.g., "alarm is off," indicating that the red alarm LED is off and entry is permitted). By Line 25 the student has taken up the reporting of the state of the system inputs and output, though the teacher is still tracking the entries and guiding the process with his finger moving to each successive row along the right side of the truth table. This is fortunate, because the student appears to be repeating the previous entry at the end of Line 25. The teacher corrects the entry (Line 25) and the student, seeing the problem, immediately initiates a repair (Line 26). In Line 28 the student notes the circuit gives the incorrect output, revealing a bug in implementing the logic electronically. In Line 29 the student notes another error. It is not until Line 29 that the teacher withdraws his finger and the student autonomously coordinates the entries of the truth table with the state of the circuit. The student then rapidly completes the coordination of the table and the circuit, glancing repeatedly between the two material forms, noting several more successes and one more error. The student then gives a summative statement of the accuracy of the circuit for implementing the logic of the monitoring system (Line 30).

Insert Case #2 Transcript 1 about here

In the language of our theoretical framework, the ***what* of mathematics** in this case is the propositional logic that instantiates the privacy conditions of the voting booth, which is reified in the truth table and in a simplified Boolean expression. The mathematics can be traced across various representational forms of the truth table (the students' preferred representation), Boolean algebra equations, the SIM diagram (the teacher's preferred representation), and the circuit configuration itself, which yields a series of outputs in the form of lighted LEDs for a given set of inputs from the switches. The teacher seeks to establish **cohesion** in several ways. The discussion of proper versus messy circuit wiring is used to illustrate how cross-modal coordination can be affected by the aesthetics of the physical implementation. This also highlights the practical consideration of constructing a well-organized breadboard to provide a clear mapping between the physical circuit and the symbolic representation of the design that is more easily traced and debugged. Cohesion is also established by showing that the trajectory of building and troubleshooting a correct circuit is not monotonic; rather, verification of the circuit involves going back to an earlier encountered representational form (in this case, truth table entries). We can see how the teacher modeled attending to the immediately present representational and material forms. Note that the teacher rarely makes explicit links between the circuit and the Boolean expression that models the context of the voting booth scenario; instead, the links are implicit in the teachers' actions. In a parallel fashion, the student's focus is specifically on coordination between truth table and circuit, rather than exhibiting the ways these forms are manifestations of logical relations that underlie the voting booth monitoring system.

Figure 5 provides an analytic view of the digital electronics case. Figure 5 shows the sequence of modal engagements as participants used Boolean algebraic expressions, an AOI diagram, a model of the circuit in MultiSims, a working electronic circuit, and engagement in the troubleshooting process. The teacher regularly used forward projection along with coordination to connect the current modal engagements and those that would be enacted at future stages of the project, striving to establish cohesion by communicating to the students the scope of the project.

The student, Steven, demonstrated learning when he appropriated the teacher's coordination-based method for verifying the accuracy of the circuit. In this way, Steven exhibited a new connection between prior modal engagements that initiated a new set of practices (Engeström, 2004), mapping directly from the tabular entries to the states of the circuit. Furthermore, the troubleshooting method was personalized when he changed the source used for comparison from the MultiSim diagram to the truth tables.

During the post-lesson interview, the teacher revealed his intentions to promote cohesion.

> Now again the big one we were talking about is going from that to the breadboard. Going from the, um, schematic to the actual… and trying to understand how it works. When they get that, then I know they've learned.

Yet he recognizes that students' typical practices can impede this understanding:

<u>Case #2 Transcript 3</u> (Some speech cut out between lines 1 and 2, only relevant gestures

1 T: Kids want a shortcut all the time, and that's what's getting them in trouble when they can't get from one [ME] to the other.

((*Teacher moves his left hand, and then his right hand*))

2 T: If they take the time to break it down they get the job done in half the time. But they want a shortcut that takes longer. Uh, and sometimes it's just a matter of just standing over them and point! Point! Point! Point!…

((*Teacher makes four dramatic pointing gestures on either side of his body, alternating between pointing with left index finger and right index finger*))

3 T: …and make sure their eyes are following.

In describing his actions, the teacher demonstrates that he is aware of the need to explicitly coordinate modal engagements for students. He is overt in his description of the indexical role that his actions can play by using his gestures and staccato speech to reenact the role of his pointing actions. In addition, he recognizes that his acts of coordination are part of a complex set of communicative interactions that are not complete unless students also fulfill their roles. By stating that he must also "make sure their eyes are following," the teacher reveals his sensitivity to the social exchange that must take place in order to establish a cohesive learning environment. In this way, the teacher presents his intent, while also tying the success of his actions to promote cohesion to the behaviors of all of the class participants.

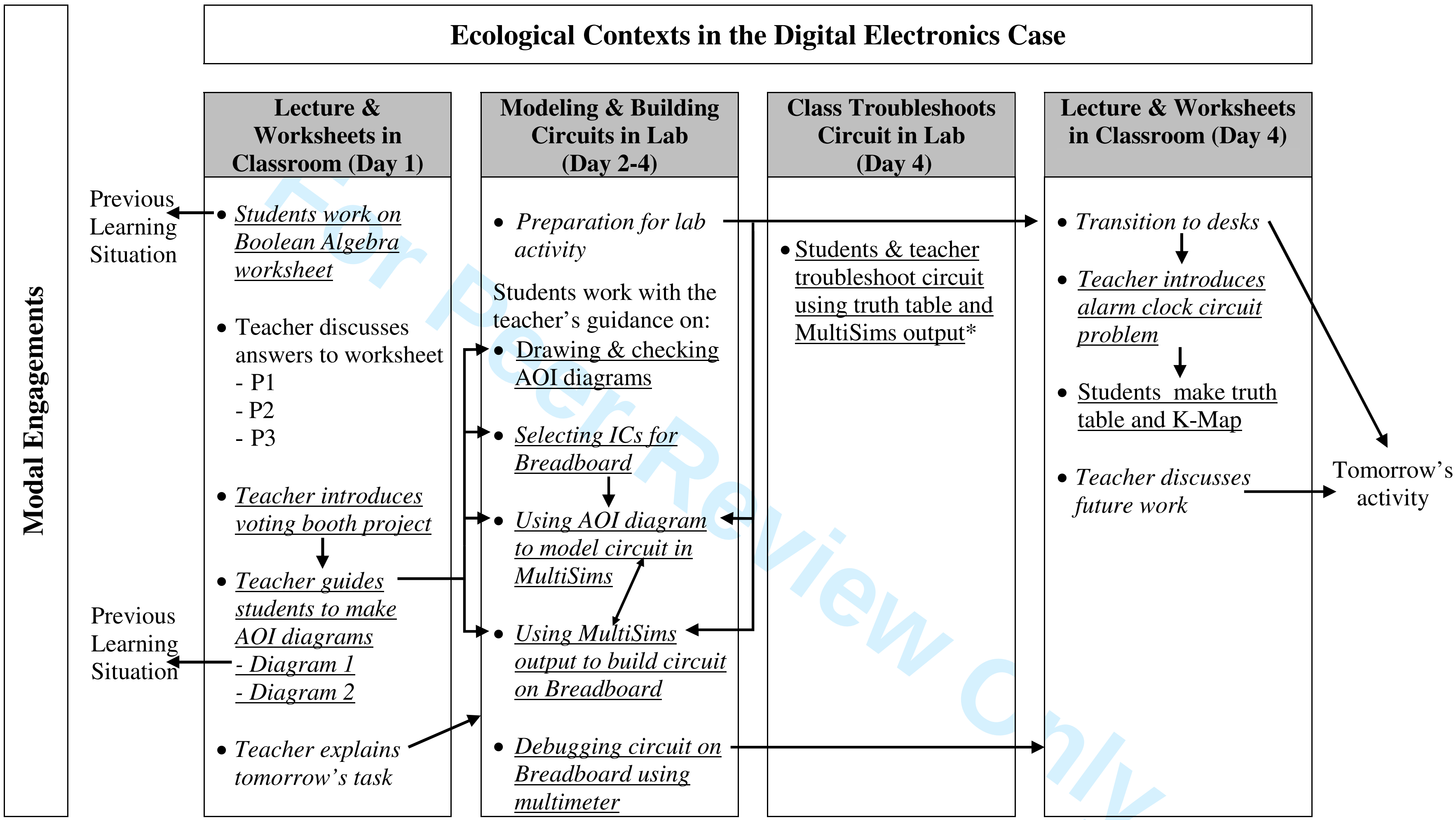


Figure 5. Modal engagements analysis of the Digital Electronics case. Columns show the major ecological contexts. Bullets show the main modal engagements. *Italics* = *Projection*, Underline = Coordination, *Italics and Underline* = *Projection + Coordination*. Arrows show the main backward/forward projections. * indicates the modal engagement discussed in the transcript.

Circles in Action: Proof in an Advanced High School Geometry Class

The final case is drawn from a high school geometry lesson about properties of circles and their associated theorems. The observed lesson alternated between three ecological contexts: individual seatwork, where a worksheet was used to practice an earlier theorem upon which the current activity builds; the computer lab, where there was access to an interactive dynamic geometry software environment; and a teacher-led discussion in the regular classroom. In the computer lab, student dyads worked together while each student had access to the program Geometer's Sketchpad (GSP). The direct manipulation interface of GSP allows students to construct, measure, and control relations between geometric objects on a computer screen. With GSP, students investigated properties of angles inscribed in circles, proposed and tested generalities, and proved (or explored) theorems through constructing, manipulating and observing different geometric figures. Hence, GSP allows students to experiment freely and to interact directly with geometrical objects and their spatial relations. In terms of proof, GSP provides a tool for students to create, validate, and refine particular conjectures through their exploration and visualization of geometry. The teacher in this setting acts as a facilitator who guides and encourages students to discover and construct knowledge for themselves.

To present this case we provide two excerpts. The first excerpt takes place in the computer lab. During this interaction, the teacher assisted several students as they generated their own explanations about why opposite angles in a quadrilateral inscribed in a circle are always supplementary (i.e., together total 180 degrees). To solve the problem, students needed to draw on an essential mathematical relation: the measure of any angle inscribed in a circle will always equal one-half of the length of its intercepted arc. This invariant relation had been encountered in a lab task and in a worksheet activity earlier that same day.

This first excerpt starts with the teacher responding to a breakdown exhibited by a student named Jordan. He appears to be stuck on the notion that the inscribed angle must intercept exactly half of the circle (as would be the case for a right angle). This idea creates a barrier for his understanding of the more general relation that the theorem would hold for inscribed angles of *any* size.

In her explanation to Jordan and other students nearby, the teacher uses coordination between the display in GSP and her gestures of the angles and arcs, along with backward projection to concepts addressed in the earlier classroom activity relating inscribed angles to the length of their intercepted arc. The teacher starts with a hint (Line 1) directing students' attention to the relevant parts of the angles and circles using speech and an explicit pointing gesture to the screen, while acknowledging (Lines 3 & 5) that some students labeled their vertices differently. In Lines 6 through 9, Jordan's response indicates a misconception that an angle and its opposing angle must each necessarily intercept a semicircle (180 degrees). Based on the theorem, this would imply that the inscribed angle could only equal ninety degrees, which is not a legitimate constraint. To begin to address Jordan's misconception, the teacher then used the dynamic nature of GSP to clarify what it means for the angle to intercept "half" a circle (Lines 9 & 10). To do so, she creates the line BE as the diagonal of rectangle ABDE on Jordan's screen and then drags Point E so that BE is positioned as a diameter (exactly half) of the circle (Line 10).

In Line 11 the teacher redirects students to the relationships more broadly. Turning to another group, she helps orient the students to the relevant parts of the circle and the angles that are inscribed in it. In Lines 11 through 18 she shows how to apply the central mathematical idea relating opposite inscribed angles of a quadrilateral to the current diagram by visually relating angle A to its intercepted arc (BD). Specifically, the teacher focuses on the angle with its vertex at A and the subtended arc that sweeps from vertex B, past C, all the way to vertex D. In Line 13 the teacher then orients the students to the angle on the opposite side of the circle from angle A, which has its vertex at C. At the end of Line 13 the teacher leaves her statement incomplete as a prompt for the student to name the arc intercepted by angle C. One student picks up on the prompt and misidentifies this as arc BCA (Line 14), surprising the teacher, who questions the response in Line 15. Her "what?" indicates this is not correct, in her view.

To further address this misstatement, the teacher becomes more concrete in Line 16, using a pointing gesture (with pen in hand as a pointer) to coordinate the term "Angle A" with a specific location on the GSP image. A student positions the mouse cursor to the same point in the diagram. In Line 18, the teacher traces with her hand along the intercepted arc from vertex B past C to D as she says "Intercepts from B to D." One of the students also traces along the arc with the mouse cursor at the same time. The teacher then directs attention to angle C in both speech and gesture, and again leaves her statement incomplete as a prompt for the subtended arc while she retracts her pen, symbolically giving the students more autonomy. A student correctly identifies the arc as BD (Line 19).

As a pedagogical move, the teacher asks if this arc BD is the same (Line 20) as the one she traced in Line 16, providing a projection back in time. A couple of students respond "no" (line 21), so the teacher scaffolds them further to consider putting the parts together (Line 22), again prompting students to name the intersected arcs on their respective screens. The response of BD (Line 23) is noncommittal, so she presses them (Line 24) to think of the parts in relation to the whole, by using two devices. First, she employs a gestural move in the air that reenacts the trace she did earlier on the computer screen (Line 16) with a counter-clockwise gesture motion in the air, again with pen in hand, that starts at Point B and traces an arc along an imaginary circle till she ends at Point D, all the while speaking about the arc BD. Then, she reenacts tracing a *clockwise* arc along the imaginary circle starting at Point B and ending at D. Finally, in one continuous gesture in the air (Line 24) she provides two complete counter-clockwise traversals of the circumference of the circle, starting and ending approximately at the location of vertex B, thereby demonstrating that the two arcs necessarily compose the whole circle. As a zoomed out version the image for Line 24 shows (Figure 6), Jordan is carefully attending to these gestures by the teacher and therefore susceptible to making this new connection.

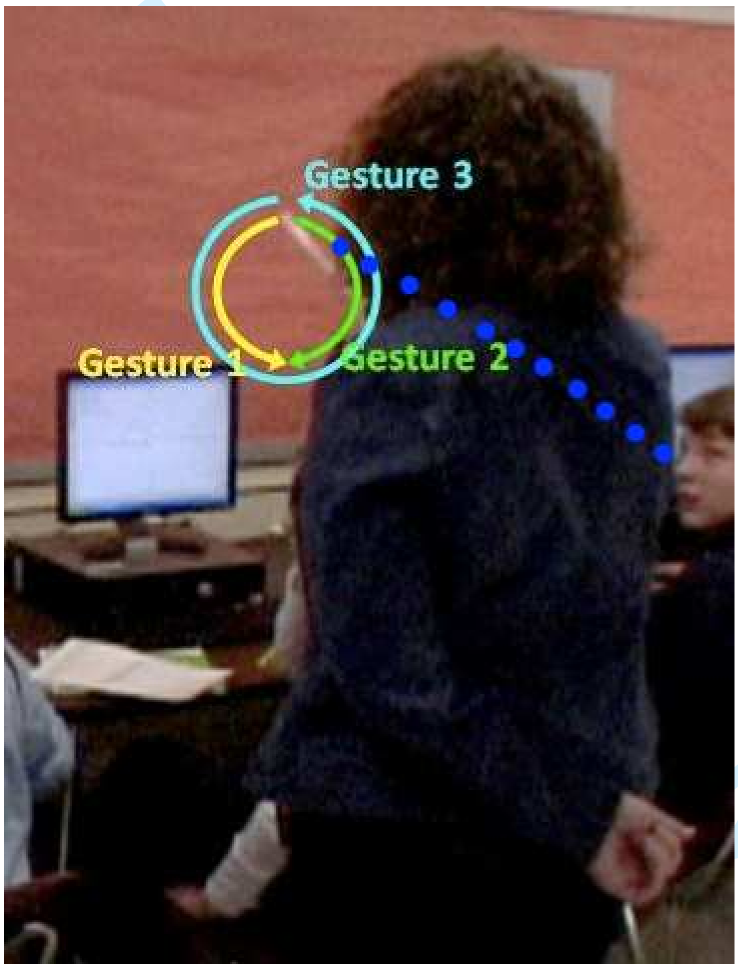


Figure 6. Zoomed out view of image from Line 24 showing Jordan attending to the teacher's circular gestures.

Following this demonstration, the teacher brings the idea back around to the language of mathematics by prompting students in Line 26 to identify the types of angles that hold this property. While a student fishes around for answers using terms from the lesson (Line 27), the teacher projects back to the previous worksheet activity that focused on inscribed angles using both speech ("on that first page…") and a gesture to an imaginary page (Line 33). The backward projection to the worksheet is a way to potentially reinstate students' knowledge of properties of inscribed angles that they had explored earlier that day (Line 34). By establishing this thread of cohesion between the earlier theorem work, the worksheet, the part-whole relations of the circles conjured by her gestures, and the GSP displays, the teacher tried to help students recognize that the two opposing angles of any inscribed rectangle are each inscribed, and therefore exhibit the invariant relation between angle and arc length that characterizes all inscribed angles.

Insert Case #3 Transcript 3a about here

The acts of coordinating the mathematical language with specific screen diagrams established cohesion between the theorem work practiced on the worksheet and the lab work, which then was connected to the teacher's actions showing how the parts sum to equal the whole circle. By repeatedly re-invoking resources like the worksheet and the teacher's gestures tracing the arcs, the teacher produced a cohesive account of how the many activities and ideas were all manifestations of the same invariant relation. Since we can observe in Figure 6 that Jordan is actually tracking the teacher's gestures used to express the conceptual relations of the arcs to the entire circumference, we may surmise that Jordan encoded this teaching moment.

Back in the classroom, removed from the computer environment, the teacher prompted students to make statements about the relations they uncovered between the inscribed angles and the arcs they intercept. In framing this session, the teacher projected backward in time to the quadrilaterals that students had inscribed in the circles, both by referring to the lab activity and by drawing a diagram resembling what students had constructed on their computer screens (Figure 7). She then coordinated the drawing with a gestural catchment – a repetition of the circular gestures she made in the lab the previous day – and thereby potentially reinvoked the GSP diagrams from the computer lab.

Figure 7. Teacher projected and coordinated the lab activity with the diagram on the whiteboard and highlighted Arcs *BAD* and *BCD* in blue (similar to her gestures in the computer lab) to help students recognize that the two arcs together form the entire circle.

Jordan now elects to speak up (Line 36) and provides direct evidence that learning has taken place. At first, he seems to be repeating his earlier misconception, when he says “we know that the inverted angle’s one half the ...” But he makes is own repair, possibly because he is using “half” in a causal way to refer to a partial amount that is one of two portions, rather than in its mathematical meaning of precisely one of two equal amounts. He also stumbles over the terminology of “inscribed angle” by saying “inverted angle.” This terminology, rather than the “half” reference, catches the teacher’s attention. In acknowledging the terminology error, he picks up his line of explanation and elaborates further. Here again (Line 40), his language seems at first to reflect the initial breakdown. Then he clarifies (Line 40) that it is not necessarily half, but "you have to figure it out," suggesting that he is aware its value is not half, but, in general, some unknown, supplementary value. This remark provides evidence that Jordan has come to recognize his own initial conceptual breakdown and has repaired it to reflect the mathematically correct conceptualization underlying the theorem. The result is that Jordan's ways of thinking about the relationship of inscribed angles to their intercepted arcs can hold for any angle.

Insert Case #3 Transcript 3b about here

Though Jordan initially showed an overly simplistic understanding of the relation of inscribed angles to their intercepted arcs (Line 6), the teacher provided support that had the potential to help him tie the theorem established earlier to the computer lab activity. With the teacher's guidance through different transitions among modal engagements, we see how Jordan was able to catch his own error and correct himself (Lines 38 and 40). He recognized that the opposite angles are each inscribed and that their intercepted arcs always form the entire circle. Eventually, having interacted with the diagrams and having witnessed the teacher's gestures of the part-whole construct both in the computer lab and the classroom, Jordan demonstrated a new understanding that opposite angles in an inscribed quadrilateral are *always* supplementary, *and* that each angle need not intersect an arc of exactly 180 degrees.

During a post-lesson interview, the teacher described her use of projection to connect students' experience in the computer lab with their mathematical explanations in the classroom.

> I think they'll understand it better 'cause you know we can refer to 'Well, remember in the lab when we did this and what did you notice?' and, you know, I

> think they're ... making those connections better than 'Oh look at your notes yesterday. What was that theorem?' (Day 2 post-lesson interview).

The teacher's use of "refer to" during the interview makes it clear that she is aware of the value of establishing these projections over time. This excerpt shows that the teacher views cohesion production as a pedagogical tool to help foster the learning of mathematical concepts across modal engagements and ecological shifts between the lab, student notes, and the classroom.

The summative analysis of the geometry case is given in Figure 8. This figure shows the cycle of activities that occur as geometry concepts are threaded through the ecological contexts of the classroom and the lab. Unlike the ballistic device and the digital electronics cases where forward projections were often employed, the geometry teacher regularly used coordination with *backward projection*, invoking the geometry relations discovered during lab activities to support the more formal discussion of concepts in classroom lectures. Such backward projections can be used to foster reflection and integrative thinking. The figure also shows the heavy use of coordination during lab activities, where participants mapped the diagrams on the computer screen to the explanations on their worksheets and to iconic gestures, realizing the core invariant relation of inscribed angles as instantiated across multiple representational forms.

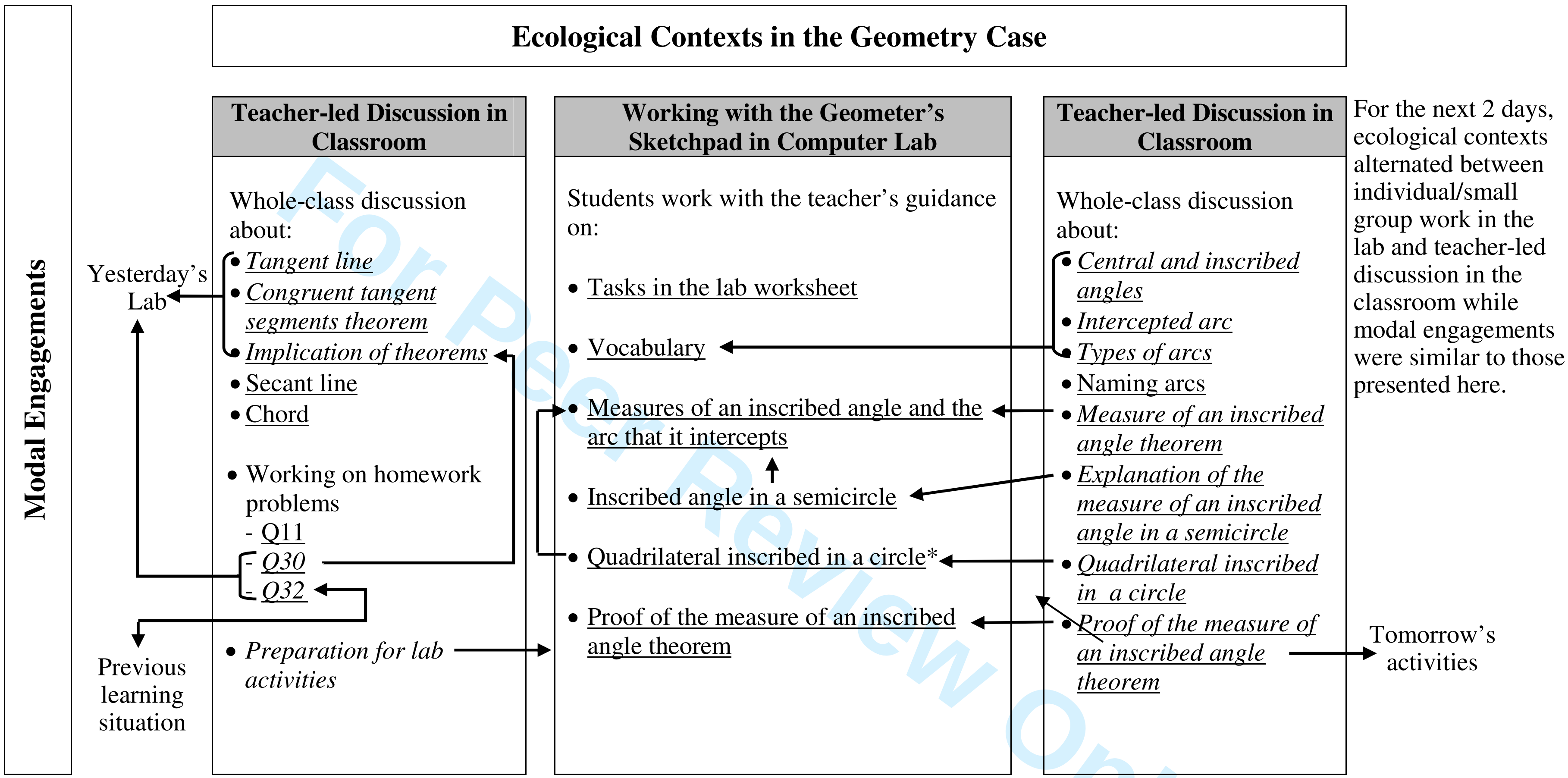


Figure 8. Modal engagements in the Geometry case. Columns show the major ecological contexts. Bullets show the main modal engagements. *Italics = Projection*, Underline = Coordination, *Italics and Underline = Projection + Coordination*. Arrows show the main backward/forward projections to past/future modal engagement(s). * indicates the modal engagement discussed in the transcript of the geometry case.

Discussion and Conclusions

In this paper, we have argued that one of the primary goals—and one of the central challenges—of STEM education is threading mathematical concepts and ideas through the various modal engagements that are commonplace in STEM disciplines. We believe that a focus on the "what" and "where" of mathematics, and on how mathematical relations connect across modal engagements and ecological contexts, can provide a valuable new lens with which to consider classroom discourse and activities. We have highlighted three transition processes that teachers (and students) use to establish and maintain cohesion: guiding attention and behavior around ecological shifts, coordinating ideas across different spaces, and projecting ideas both forward and backward in time. As we have demonstrated in post-lesson interviews, teachers intentionally use these approaches to support student understanding. Furthermore, when students are attentive to such cohesion-producing moves, those moves can engender learning by highlighting new connections, like those Jordan made in reconceptualizing inscribed angles in the geometry case, and by affording new, complex practices, such as Steven's method of troubleshooting the digital circuit.

Of course, several features of the current research limit the extent to which one can generalize the results. The limited sample size and the particulars of the settings and participants call for treating the findings provisionally, pending further corroboration across greater numbers of students in a broader array of learning environments. Future studies should more directly evaluate the impact of cohesion production on student outcome measures such as learning, knowledge organization and transfer to other classroom tasks and to other STEM domains. In this regard, our findings abut teachers' use of projection are reminiscent of recent work on "expansive" framing of learning contexts, which has been shown to promote transfer (Engle, Nguyen & Mendelson, 2011). Teachers' projections to past and future instantiations of a concept help establish the "temporal horizon" (Engle et al., p. 610) of a lesson. We hypothesize that greater use of projection should contribute to a more expansive framing of the lesson, which in turn should promote transfer.

In this final section we consider some of the other broad issues that this work touches upon; specifically, connecting abstract and concrete ideas during instruction, the modal nature of mathematics, and the relation of being and learning.

Connecting Abstract and Concrete Ideas

Cohesion as it is employed here is both related to and distinct from the notion of *grounding,* which refers to connecting more abstract and unfamiliar concepts and ideas to more familiar and concrete ones (Clark & Brennan, 1991; Harnad, 1990; Koedinger, Alibali, & Nathan, 2008). The notion of grounding is often invoked in project-based learning and reform approaches to education, on the argument that context, materials and activity structures -- modal engagements that commonly occur in STEM classrooms -- help to establish the meaning and appropriate uses of abstract ideas in concrete and familiar ways, and help to make schooling more relevant (Blumenfeld et al., 1991; Hmelo, Holton & Kolodner, 2000; Jurow, 2005). Yet we observed in these three cases that grounding contexts and activities cannot be assumed to enhance understanding and learning. Indeed, tracking invariant relations across their many manifestations appears to be a challenge for many students operating in rich, multimodal learning environments. This is because grounding contexts also introduce new demands for establishing cohesion across the familiar and new modal engagements. In order to realize the potential of grounding as a means to facilitate understanding, learners must grasp the relation of the mathematics to the grounding objects and activities themselves. Teachers and curriculum developers can improve students' prospects when they explicitly attend to these links.

It is worth considering how the connections between abstract and concrete ideas play out in STEM classrooms. In both of the engineering lessons we observed (projectile motion and digital electronics), there is a general sense that the lessons move from abstract symbolic systems of notation (e.g., Boolean expressions and algebraic equations) to more concrete modal engagements (e.g., working mechanical and electronic devices). In contrast, in the geometry class, the opposite trajectory was observed. The lesson started out concrete (with observations of the behavior of actual angles and circles) and moved to the abstract (formal theorems). Of course, it is clearly not warranted to draw inferences from such a small sample of classes, but we note that these patterns are generally consistent with the different orientations that have often been ascribed, respectively, to college preparatory classes (geometry) and technical education programs (Rose, 2004).

The Modal Nature of Mathematics

Data of the sort we offer here illustrate that mathematics is not an amodal or purely "formal system," as its expression in symbolic form is but one of many ways that mathematical ideas are realized. Furthermore, each observed modal engagement in these cases tended to occur with its own patterns of verbal discourse, gestures, form-specific operations, manipulations, and perceptions; each idea tended to have a particular "nesting ground," where it was most likely to be found, especially early in its inception; and each curriculum unit developed a particular historical trace between general (abstract) representations and particular (concrete) objects and events, though there were also instances of forward and backward projections.

From a pedagogical point of view, this work suggests a need for teachers to reframe math more broadly for learners, as well as curriculum developers and assessment designers, by vastly extending the range of mathematical representations and cross-representational mappings under consideration. In this research, we have sought to expand the discussion in STEM education about what constitutes mathematical activity and what constitutes mathematical representations.

Expanding our view of what constitutes mathematical representations raises new challenges for understanding students' acquisition of *representational fluency*, which we define as the ability to work with and translate among multiple representations (Nathan et al., 2002). Representational fluency is of central importance to scientific and mathematical performance, as each representation form offers distinct ways of realizing the target mathematical ideas and each enables and privileges certain methods of communication about those ideas, while de-emphasizing others. When interacting with representations, students need to have the abilities to understand, select, construct, and effectively use different representational forms to make sense of their learning experiences (diSessa, 2004). We hypothesize that teachers' efforts to establish cohesion are an important contributor to students' acquisition of the abilities that mediate representational fluency.

Being and Learning in STEM Classrooms

One of the central issues to emerge from the analysis of cohesion production in the classroom is a greater appreciation of the challenges of STEM integration from the learners' perspective. There is a tendency to see hands-on activities and authentic contexts as powerful ways to ground new ideas and abstract representations. The analyses underscore the novel demands of working in multi-modal learning environments. Why should this be so?

The philosopher Martin Heidegger argued that "skillful coping," not theoretically driven action and critical reflection, characterizes the vast majority of our everyday behaviors (Dreyfus, 1991). When we eat, build something, or drive a car, we do not continually experience the world mediated through mental representations, but instead we act directly upon the world through our being (or *Dasein*) by employing "background practices" in the manners in which we were socialized to act. Implements of our functioning (forks, cars) are invisible to us most of the time, just as, when nailing boards, the skilled carpenter is aware but not self-aware of the hammer.

Sometimes, Heidegger concedes, coping is insufficient. One circumstance in which coping is inadequate is when the world (or tool) "breaks" (Dreyfus, 1991, p. 4). At such moments we do engage in critical reflection and intentional thinking mediated by representations of the world. This is not only true of students' individual experiences but is also found when professionals working in teams that cross traditional disciplinary boundaries encounter "disruptions" in their work flow and the specifics of the underlying representational infrastructures become apparent and open to reformulation (Hall, Stevens, & Torralba, 2002).

"Coping" may also be inadequate when people are engaged in science and math education where principle-driven reasoning and critical reflection are central objectives. Little in the way of learning can be expected from coping; rather it is during critical reflection on the mediating representations that self-regulated learning occurs.

On their own, students tended to exhibit the coping behavior of everyday practices that Heidegger argued must necessarily be antecedent to mental representations. To achieve the lofty objectives of STEM education, students—and all learners—must be lulled out of their normal patterns. Educators strive to engineer disruptions of coping behaviors and create "breaks" in the environment (Nathan & Kim, 2009), thereby showing how everyday practices can fall short or are simply incapable for addressing some scientific matters. In place of coping, learners may adopt a new stance, one that enables the production of analytical and propositional thinking that is commensurate with cognitive accounts of scientific reasoning. In the present work, we saw this most vividly in the geometry case, when Jordan corrects his own misunderstanding and provides the more accurate, more general and more nuanced account relating inscribed angles. Threading through, then, may be characterized as more than an approach that is pedagogical or even epistemological, but rather one that is ontological, transforming the very manner in which learners are *being* so they align, albeit temporarily, with scientific modes of practice.

Conclusion

We have argued that one of the central challenges of STEM education is threading mathematical concepts and ideas through the various modal engagements that are commonplace in STEM disciplines in order to make the learning environment more cohesive. We believe that this focus on the "what" and "where" of mathematics provides a valuable new lens with which to consider classroom discourse and activities, and also teachers' intentions and goals. This view also provides a new perspective on student learning and on the conditions and pedagogy that may best foster that learning. A focus on cohesion—including how it is implemented in teaching, and how it affects learning—can provide a new framework for considering and addressing the challenges that teachers and students face in STEM education every day.

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Appendix A:

An Example of Two Different Physical Systems that Share Common Mathematical Models

Consider the series RLC circuit shown in Figure A-1, with resistance R (measured in Ohms), inductance L (Henrys), capacitance C (Farads), and voltage across the battery V (Volts). We can use the voltage equations for each circuit element and Kirchoff's voltage law to write a second order linear constant coefficient differential equation (Eqn. 1) describing the charge on the capacitor over time (q(t)). An analogous model (Eqn. 2) can be used for the mechanical system in Figure A-2, which shows displacement as a function of time (x(t)) when a force F (Newtons) is applied to an ideal mass-spring-damper circuit with mass m (kg), spring constant k (N/m) and damping coefficient R (N-s/m).

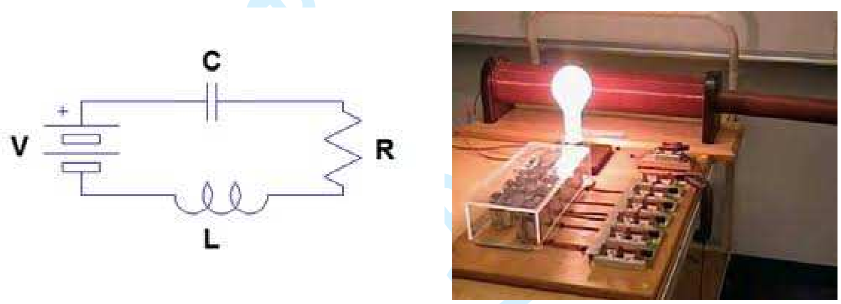


Figure A-1. A series resistor-inductor-capacitor (RLC) circuit and a real life set up.

(Eqn. 1) $$L\, q(t)'' + R\, q(t)' + (1/C)q = V$$

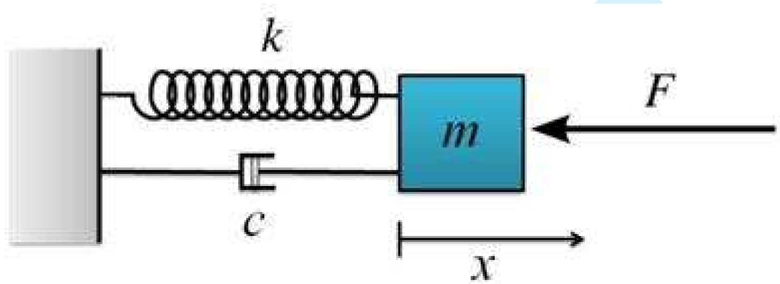


Figure A-2. A series mass-spring-damper circuit.

(Eqn. 2) $$mx(t)'' + Rx(t)' + kx = F$$

Case #1 Transcript 1.

(NB. Speech transcript is complete but only gestures relevant to this analysis are shown. Square boxes denote the start and end of the gesture, red arrows indicate direction of movement, and green arrows indicate the location of pointing)

1 T: Well I'm wondering if the further you, pull your rubber band down-

2 S Mmhm.

3 T: is gonna affect your, velocity, more than your angle.

((Teacher points to diagram))

4 S: Yeah it's. Well no this is the velocity but what we're sayin' is that this is how hard it pulls

((Student points fingers at different parts of diagram))

5 S: but then right here, where it where it…

((Student makes flat-hand gesture on top of diagram))

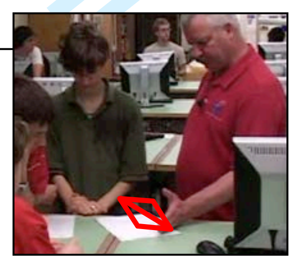
6 S: where the fulcrum is like this actually you can tilt it.

((Student makes arm into lever))

7 S: The rubber bands control the tension but the
placement is what really controls...

8 S: Like. See what we're saying?

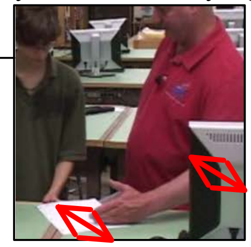
9 T: So it's it okay so, if I could, suggest, I think that, you might be able to adjust your angle
by, by having some type, by controlling where this stops.



((Teacher positions flat hand over diagram, and moves fingertips up and down while keeping base of hand stationary))

10 S: Yeah.

11 T: But that's probably also gonna affect your, maybe affect your velocity. What I'm saying is. Either that or else you have to tip the whole thing.



((Teacher places flat hand over diagram, this time moving both ends of his arm back and forth))

12 S: No we don't. That's why cuz the two sides stay put but then the top part can, tilt…

((Student places to flat hands parallel to each other, and then places flat hand perpendicular to other hand, tilting palm upwards and downards))

13 S: right there.

((Student points to diagram))

14 T: Okay.

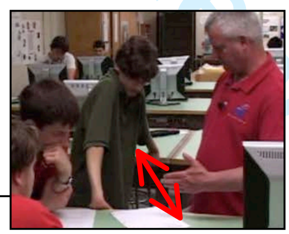
15 S: So the fulcrum can change positions basically.

((Student traces back and forth on table with finger))

16 T: Alright. So I think maybe what you need to do is is, take into consideration what I just said about-

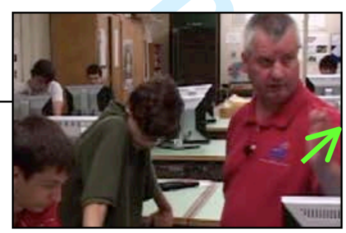
17 S: Yeah.

18 T: being able to control the ang- …



((Teacher moves flat palm back and forth in the air))

19 T: that's why we did everything we did here



((Teacher points to board))

20 S: Mmhm.

21 T: -with the math. Because we wanna-

22 S: the math yeah.

23 T: -be able to adjust the angle of the trajectory. I would try to keep, the velocity, the same, consistent, throughout the whole every test that you do that that's consistent and so all you're gonna change once you once you decide what that velocity has to be all you're gonna change is your angle.

24 S: Yeah.

25 T: Okay?

26 S: Mmhm.

27 T: I don't really want you to use the tension on the rubber bands, as, the only control. I want you to have an angle adjustment.

<u>Case #2 Transcript 2</u> (NB. Speech transcript is complete but only gestures relevant to this analysis are shown. Square boxes denote the start and end of the gesture, red arrows indicate direction of movement, and green arrows indicate the location of pointing)

1 T: Guys everybody stop come over here ’cause we’re gonna stop here and then we’re gonna do an exercise, up in the front. So but I need you everybody stop what you're doing leave it come over here. (Name) come on. Okay. It says ninety percent working but I want to make some comments about the board.

2 Steven: Yeah it’s messy. I get it.

3 T: I don’t have to make comments about the board you just did it.

4 Steven: Yeah.

5 T: Right? What’s uh the term I’m always giving you is spaghetti.

6 Steven: Spaghetti.

7 T: To try to solve problems and you got stuff running all over it’s much harder to do but I’m glad for the most part you’ve got it working. So just demonstrate to me that what you’ve got working but you need to put your wires-

8 Steven: I just need-

9 T: -so they-

10 Steven: -to put the switch.

11 T: -they’re not at angles try to get them all square so you can follow a path, laying right next to each other. Nothing goes over switches, nothing goes over the integrated circuits, get ’em straight, and if you got a long wire and you’ve got to make a bubble out of it shorten

the wire. And I'm always saying if you have like these here are going at angles those could have been shortened straightened out. Kay.

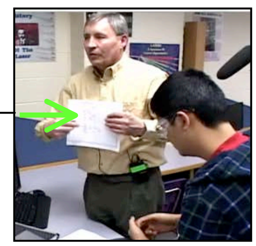
12 Steven: Oh yeah.

13 T: And on your paperwork when you're doing the check, you have numbers and letters here. What hole is that in? There's a l- number and a letter. Use 'em

((Teacher points to two ends of breadboard))

14 T: And to check things off. Write 'em right on here. I did this one, this one's hooked up, go to the next one, look, put the number on here.

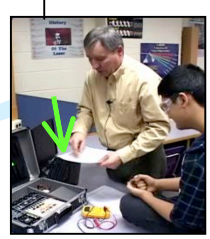
((Teacher points to SIM diagram))

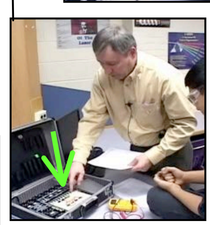


15 T: You know 1A. You know is it 10B. What are the things plugged into? Well that's your checklist. Otherwise it's hard just look at this as a whole picture, it's just the spaghetti mess.

((Teacher points to position on breadboard, then indicates SIM diagram))

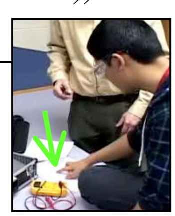
16 T: But uh now I can follow this. If I know that you want to do something I can look. Look at the number and say oop you're in Hole 2 when it should be Hole 3. You just put it in the wrong hole and that's your troubleshooting. That's a checklist by putting it on here. Alright go through and show me what does work.





((Teacher points to SIM diagram and then points to breadboard))

17 Steven: Oh okay, we'll uh we'll just go with this thing.



((Steven points to truth table))

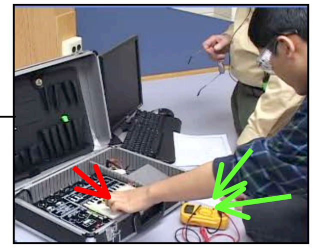
18 T: Alright.

19 Steven: Okay uh.

20 T: So we...

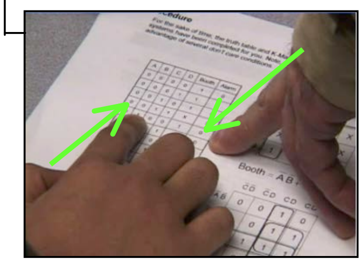
21 Steven: Booth alarm all of this is…

((Steven points to one end of breadboard))



22 Steven: (at the same time) off right now?

T: (at the same time) Kay so you're doing zero zero zero.



((Steven flips switches on breadboard and points to truth table with two different hands, while teacher points to truth table only. These gestures continue through the end of line 28))

23 Steven: So zero zero zero booth is on alarm is off.

24 T: Okay.

25 Steven: Zero zero one booth is on alarm is off. Uh zero zero one (at the same time)...

T: (at the same time) Zero one.

26 Steven: zero booth alarm same thing. This too so the green one should come on here and it does and the red one doesn't matter.

27 T: Yeah.

28 Steven: Uhhh (pause) yeah green one's okay, so so far it works. Oh see that one's the one that oh so the green one doesn't work but the red one works for that one over there. I'll keep that a mental note okay.

29 Steven: Uh so these two doesn't work uh the third one eh this one works. The thir- that one works. Uhhh (pause) okay this

one should be off but yeah. Uh this one works. That one works. That one works. This uh that one works. That one works. And that one works.

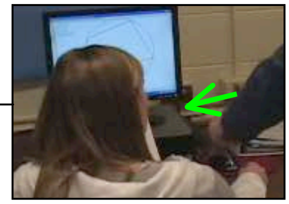
((Teacher removes finger from truth table and Steven continues flipping switches on breadboard and pointing to truth table throughout Line 29))

30 Steven: So there's three doesn't work.

Case #3 Transcript 3a (NB. Speech transcript is complete but only relevant gestures are shown.)

1 T: Okay so here's my hint. Look at angle A. What arc does- well even on your picture there, well you didn't label- okay well

look at (Name)'s angle A over here on her picture.



((*Teacher points with index finger to student's worksheet*))

What arc does that intercept? Angle A should intercept- oh you've got an E there huh?

2 Jordan: Yeah

3 T: For you it's BE and for these guys it's BD. And now look at the angle across from A having the opposite angle, right?

4 S: Mmhm.

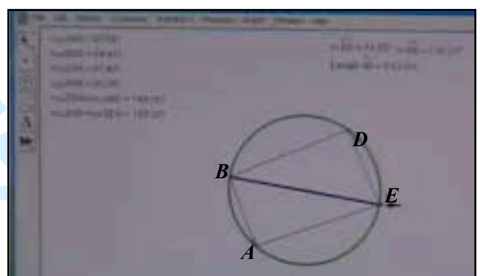
5 T: In yours that's angle C.

6 Jordan: Oh, it intercept-s- it intercepts the other half.

7 T: What do you mean the other half?

8 Jordan: Okay well.

9 T: These are half?



10 Group: (* loud background chatter, inaudible *)

(At the same time) (* Teacher works individually with Jordan on his screen

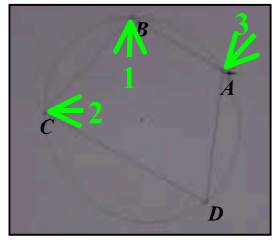
((*Teacher creates $\overline{BE}$ on Jordan's sketch and drags Point E so that $\overline{BE}$ is a diameter*))

before coming back to work with the whole group *)

11 T: ((*To a different student with rectangle ABCD)*) So, wait, angle A intercepts the arc from B all the way around to D, right?

12 S: Yeah.

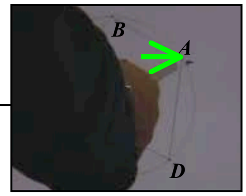
13 T: And the angle across from that, angle C, intercepts …



14 S: B (pause) CA

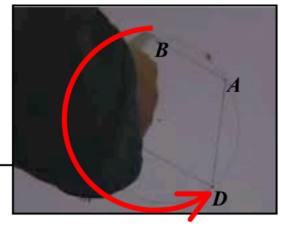
((*Student moves the pointer on her sketch to Points B, C, and A*))

15 T: What?



16 T: Angle A, right?

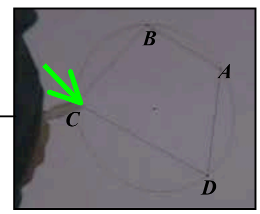
((*Teacher points with a pen to Point A while Student moves her pointer to the same position*))



17 Jordan: Intercepts B-

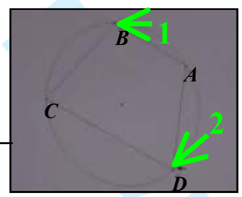
18 T: Intercepts from B to D.

((*Teacher sweeps pen from Point B to Point D toward Point C while Student tracks her pen with the pointer*))



Angle C intercepts …

((*Teacher points with pen to Point C*))



19 S: BD

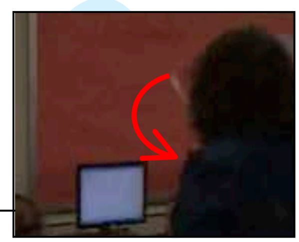
((*Student moves the pointer to Point B and then Point D*))

20 T: Is it the same BD?

21 Jordan and S: No.

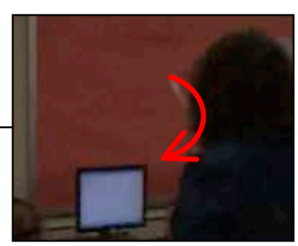
22 T: No. Together those two angles intercept … what …

23 S: BD.



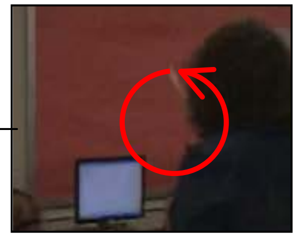
24 T: But it's BD on this part

((*Teacher draws a left arc in the air with pen as though tracing a portion of an imaginary circle*))



and BD on this part, which is?

((*Teacher draws an arc with pen on the right side *of the imaginary circle in the air*))



The entire circle right?

((*Teacher forms in the air a complete circle with the pen starting approximately where Point B is located near the top of the imaginary circle*))

25 S: Yeah.

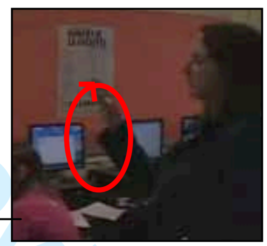
26 T: Cool. And so then what does that mean about those angles? What kind of angles are they?

27 S: Supplementary.

28 T: Why?

29 S: ’Cause.

30 Jordan: They are.



31 T: What kind of angles are they?

32 S: (* talking inaudibly *)

33 T: Okay they are opposite, but in the circle?

((*Teacher moves her hand in circles*))

On that first page you talked about-

((*Teacher points with finger to student’s worksheet*))

34 S: Inscribed.

35 T: So think about what you know about inscribed angles. Alright I think you’re almost there.

The following conversation between the teacher and Jordan shows that the teacher’s efforts paid off. Here Jordan, while still struggling to select the proper mathematical terminology, exhibits understanding of the central mathematical relationship of the earlier lesson.

Case #3 Transcript 3b

((Students and the teacher are now back in the classroom, with a teacher led discussion using the whiteboard.))

36 Jordan: Okay so since we know a circle's three hundred-sixty degrees and if those two angles take up an entire circle and we know that the inverted angle's one half the-

37 T: Inverted angle? Inscribed angle?

38 Jordan: Yeah. Yeah.

39 T: Okay.

40 Jordan: You all know what I'm talkin' about. That would be three hundred and sixty divided by two divided by two. Because you have- well divided by two and then you have to figure it out.